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BOND PRICING WITH TERM STRUCTURE MODELS: SOME EMPIRICAL EVIDENCE

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Abstract

We compare the Longstaff and Schwartz (1992) (LS) two-factor general equilibrium model with the Schwartz (1984) (SS) two-factor arbitrage model of the term structure of interest rates. The Cox, Ingersoll, and Ross (1985) (CIR) one-factor model is also studied as a reference. LS use as state variables the short term interest rate and the volatility of the short-term interest rate, while SS use the spread between the short-term and the long-term interest rate, and the long-term interest rate. If the general equilibrium approach and the arbitrage approach are equivalent, the LS model should perform better (worse) than the SS model in pricing short-term (long-term) securities. Moreover, since the CIR model can be nested into the LS model, it is expected that the latter model perform better than the former one.

The results show that, as expected, the LS model is best when pricing short-term discount bonds, while the SS model is best when pricing long-term bonds. However, both models have difficulties adjusting to the term structure of interest rates. This problem is more evident in the CIR model.

Keywords
Term structure, bond pricing

Bond Pricing with Term Structure Models: Some Empirical Evidence

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Abstract

In this study we compare the Longstaff and Schwartz (1992) (LS) two-factor general equilibrium model with the Schaefer and Schwartz (1984) (SS) two-factor arbitrage model of the term structure of interest rates. The Cox, Ingersoll, and Ross (1985b) (CIR) one-factor model is also studied as a reference. LS use as state variables the short term interest rate and the volatility of the short-term interest rate, while SS use the spread between the short-term and the long-term interest rate, and the long-term interest rate. If the general equilibrium approach and the arbitrage approach are equivalent, the LS model should perform better (worse) than the SS model in pricing short-term (long-term) securities. Moreover, since the CIR model can be nested into the LS model, it is expected that the latter model perform better than the former one.

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1 Introduction

There are three approaches to price contingent claims when the evolution of interest rates is stochastic. The arbitrage approach derives a partial differential equation (PDE) for the value of any contingent claim by constructing portfolios of securities whose weights are chosen to make the rate of return on the portfolio non-stochastic. Then, to avoid the possibility of arbitrage profits, the rate of return on the portfolio is made equal to the instantaneous riskless rate of interest. Examples of this method are the one-factor models of Merton (1973), Brennan and Schwartz (1977), Vasicek (1977) and Dothan (1978); and the two-factor models of Brennan and Schwartz (1979 & 1982), Schaefer and Schwartz (1984) (SS), and Moreno (1996).

The general equilibrium approach, developed by Cox, Ingersoll, and Ross (1981), (1985a) and (1985b), uses an intertemporal general equilibrium asset pricing model to study the term structure of interest rates. In their model, the pricing equations incorporate anticipations, risk aversion, investment alternatives, and preferences about the timing of consumption, and asset prices and their stochastic properties are determined endogenously. The equilibrium price of any asset is given in terms of the underlying real variables in the economy and it is consistent with maximizing behavior and rational expectations. As an application, Cox, Ingersoll, and Ross (1985b) (CIR hereafter), develop a one-factor model of the term structure and use it to price bonds and bond options. In this model the term structure of interest rates at any point in time is given by the current level of short-term interest rate, and the volatility of the interest rate process is proportional to the level of the short-term interest rate. A two-factor term structure model using the general equilibrium approach can be found in Longstaff and Schwartz (1992)(LS). They use the short-term interest rate and its instantaneous variance as state variables, and they derive closed-form expressions for the price of discount bonds and discount bond options. The authors find empirical support for their model.

All these models imply term structures of interest rates that, generally, are not consistent with the market yield curve. Thus, the models will price incorrectly the underlying assets of most interest rate derivatives. As a consequence, a third kind of models, consistent with the current term structure of interest rates, are constructed. This is the approach followed by Ho and Lee (1986), using bond prices, and Heath, Jarrow, and Morton (1992), using forward interest rates. An equivalent technique is to extend non-consistent term structure models, using time-depending parameters that are calibrated to make the model match the market yield curve (see, for example, Hull and White (1990a), Black, Derman, and Toy (1990), and Black and Karasinski (1991)).

Alternative derivative models can be compared in two ways. First, we can analyze the ability of their state variable processes to describe the evolution of interest rates. For example, Chan, Karolyi, Longstaff, and Sanders (1992) use the Generalized Method of Moments of Hansen (1982) as a general framework to estimate and compare one-factor models of the short-term interest rate. They find that the most successful models in capturing the dynamics of interest rate are those that allow the volatility of interest rate changes to be highly sensitive to the level of the riskless rate[1]. The comparison of two-factor models has received much less attention in the literature[2], perhaps because of the absence of a general econometric framework in which these models can be easily nested.

However, a model can be successful at describing the evolution of a particular interest rate, but it can perform poorly when pricing contingent claims. This leads us to a second, and maybe more convenient, way of comparing alternative models: to study the performance of the models when pricing securities. See, for example, Bühler et al. (1999), Navas (1999), and Moraleda and Pelsser (2000).

The purpose of this paper is to compare models of the term structure in terms of

[1] A similar result is obtained by Navas (1999) in the Spanish market.

[2] Some exceptions are Dai and Singleton (2000) and Bühler, Uhrig, Walter, and Weber (1999).

their ability to price bonds. Specifically, we compare the pricing of pure discount bonds of the two-factor general equilibrium model of Longstaff and Schwartz (1992) and the two-factor arbitrage approach of Schaefer and Schwartz (1984). As additional reference, the single-factor CIR model is also used in the comparisons.

The reason for concentrating on two-factor models is that, as LS argue, single-factor models imply that the instantaneous returns on bonds of all maturities are perfectly correlated, which is not supported empirically. Moreover, the reason for choosing these particular two-factor models is that they have closed-form solutions for bond prices[3,4].

The remainder of the paper is organized as follows. Section 2 presents the models under consideration. Section 3 describes the estimation and implementation of the models. Section 4 studies the performance of the models when pricing discount bonds. Finally, Section 5 summarizes the paper.

2 The Models

Term structure models are developed assuming stochastic processes for the dynamics of one or more exogenous factors or state variables. Typically, single-factor models use different specifications of the short-term interest rate process and assume that the term structure of interest rates is a function of the level of the current riskless rate (and some parameters). Two-factor models assume that the term structure is a function of the current values of two state variables (frequently including the short-term interest rate).

Then, the general equilibrium or the arbitrage approach can be used to derive a partial differential equation (PDE) for the value of any contingent claim. In this equation the coefficients of the variables are functions of the parameters of the processes describ-

[3] Although in the SS model the solution is only an approximation.

[4] Balduzzi, Das, Foresi, and Sundaram (1996) and Moreno (1996) also provide closed-form expressions.

ing the term structure and of the market prices of the risks associated with the state variables. Models based on the arbitrage approach make explicit assumptions about the functional forms of these market prices of risk, making it possible in some cases to reduce the number of unknown parameters to be estimated (see, for example, Brennan and Schwartz (1979) and Schaefer and Schwartz (1984)). In these models the functional forms of the market prices of risk may not be consistent with market data. This problem is avoided when the general equilibrium approach is used, since, in this case, the functional forms of the market prices of risk are obtained endogenously, ensuring that the risk premium is consistent with no arbitrage.

The pricing equation must be satisfied for the value of any security. Depending on the asset being priced, this PDE will have different boundary conditions and, therefore, different solutions. In general, closed-form solutions to the pricing equation are not known, and numerical procedures are required (see Brennan and Schwartz (1978), Courtadon (1982), Geske and Shastri (1985), and Hull and White (1990b), among others).

2.1 The Longstaff and Schwartz (1992) model

The state variables in this model are the short-term interest rate, r, and the instantaneous variance of changes in the short-term interest rate, V. Using general equilibrium considerations, LS show that the dynamics of r and V are given by

$$\begin{aligned} dr &= \left(\alpha\gamma + \beta\eta - \frac{\beta\delta - \alpha\xi}{\beta - \alpha} r - \frac{\xi - \delta}{\beta - \alpha} V\right) dt \\ &\quad + \alpha\sqrt{\frac{\beta r - V}{\alpha(\beta - \alpha)}} dz_1 + \beta\sqrt{\frac{V - \alpha r}{\beta(\beta - \alpha)}} dz_2, \qquad (1) \\ dV &= \left(\alpha^2\gamma + \beta^2\eta - \frac{\alpha\beta(\delta - \xi)}{\beta - \alpha} r - \frac{\beta\xi - \alpha\delta}{\beta - \alpha} V\right) dt \\ &\quad + \alpha^2\sqrt{\frac{\beta r - V}{\alpha(\beta - \alpha)}} dz_1 + \beta^2\sqrt{\frac{V - \alpha r}{\beta(\beta - \alpha)}} dz_2. \qquad (2) \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \eta$, and ξ are constants, and z_1, and z_2 are uncorrelated standard Wiener processes.

If $P(r, V, \tau)$ is the value of a pure discount bond with maturity τ, LS apply Theorem 3 of Cox, Ingersoll, and Ross (1985a) and derive a PDE for P, with boundary condition $P(r, V, 0) = 1$. Making the corresponding change of variables, the solution to this equation is given by

$$P(r, V, \tau) = A^{2\nu}(\tau) B^{2\eta}(\tau) \exp(\pi\tau + C(\tau)r + D(\tau)V) \tag{3}$$

where

$$\begin{aligned}
A(\tau) &= \frac{2\phi}{(\delta + \phi)\left(e^{\phi\tau} - 1\right) + 2\phi} \\
B(\tau) &= \frac{2\psi}{(\nu + \psi)\left(e^{\psi\tau} - 1\right) + 2\psi} \\
C(\tau) &= \frac{\alpha\phi\left(e^{\psi\tau} - 1\right) B(\tau) - \beta\psi\left(e^{\phi\tau} - 1\right) A(\tau)}{\phi\psi(\beta - \alpha)} \\
D(\tau) &= \frac{\psi\left(e^{\phi\tau} - 1\right) A(\tau) - \phi\left(e^{\psi\tau} - 1\right) B(\tau)}{\phi\psi(\beta - \alpha)}
\end{aligned}$$

and

$$\begin{aligned}
\nu &= \xi + \lambda, \\
\phi &= \sqrt{2\alpha + \delta^2}, \\
\psi &= \sqrt{2\beta + \nu^2}, \\
\pi &= \gamma(\delta + \phi) + \eta(\nu + \psi).
\end{aligned}$$

Thus, the discount bond price is a function of r, V, and τ, and depends on the parameters $\alpha, \beta, \delta, \gamma, \eta$, and ξ, as well as the market price of risk λ (assumed to be constant through time). Finally, the yield to maturity can easily be obtained as follows

$$Y(r, V, \tau) = -\frac{\ln P(r, V, \tau)}{\tau}. \tag{4}$$

2.2 The Schaefer and Schwartz (1984) model

In this model the state variables are the spread between the short-term and the long-term interest rates, s, and the long term interest rate, l. The dynamics of these variables are assumed to be given by the following stochastic differential equations

$$ds = m(\mu - s)dt + \gamma dz_1 \tag{5}$$

$$dl = \beta(s, l, t)dt + \sigma\sqrt{l}dz_2, \tag{6}$$

where m, μ, γ, and σ are constants, and z_1, and z_2 are standard Wiener processes, with instantaneous correlation ρ.

Thus, the spread is modeled as a Ornstein-Uhlenbeck process with constant volatility. However, the volatility of the long-term interest rate is assumed to be proportional to its level.

SS leave the drift of the long-interest rate process, $\beta(s, l, t)$, unspecified since any drift is compatible with their pricing equation. In this paper, we assume that there is mean reversion in long-term interest rates, so that this rate follows the square root process

$$dl = \kappa(\theta - l)dt + \sigma\sqrt{l}dz_2 \tag{7}$$

Assuming that the market price of spread risk, λ, is constant, and that the spread is uncorrelated with the long-term rate, i.e. $\rho = 0$ (consistent with the empirical evidence[5]), Schaefer and Schwartz (1984) show that the value of a default free bond, P, must satisfy the PDE

$$\frac{1}{2}\gamma^2 P_{ss} + \frac{1}{2}\sigma^2 l P_{ll} + P_s m(\hat{\mu} - s) + P_l\left(\sigma^2 - ls\right) - (l+s)P - P_\tau = 0, \tag{8}$$

with the boundary condition $P(s, l, 0) = 1$, where $\hat{\mu} = \mu - \frac{\lambda\gamma}{m}$.

An approximated solution to this equation is given by

$$P(s, l, \tau) = X(s, \tau)A(\tau)\exp(-B(\tau)l), \tag{9}$$

where

$$\begin{aligned}
X(s,\tau) &= \exp\left[\frac{1}{m}\left(1 - e^{-m\tau}\right)(s_\infty - s) - \tau s_\infty - \frac{\gamma^2}{4m^3}\left(1 - e^{-m\tau}\right)^2\right], \\
A(\tau) &= \left(\frac{2\alpha\exp\left((\hat{s}+\alpha)\tau/2\right)}{(\hat{s}+\alpha)(\exp(\alpha\tau)-1)+2\alpha}\right)^2, \\
B(\tau) &= \frac{2(\exp(\alpha\tau)-1)}{(\hat{s}+\alpha)(\exp(\alpha\tau)-1)+2\alpha}, \\
s_\infty &= \hat{\mu} - \frac{\gamma^2}{2m^2}, \\
\alpha &= \sqrt{\hat{s}^2 + 2\sigma^2}.
\end{aligned}$$

Here, $\hat{s}$ depends on the current values of s and l, as shown by SS, and it is obtained numerically equating messy nonlinear functions of $\hat{s}$. However, when the initial value of the spread (s_0) is equal to $\hat{\mu}$, then $\hat{s}$ is also equal to $\hat{\mu}$.

Using the bond price, we compute the yield to maturity as in expression (4).

[5] See, for example, Ayres and Barry (1980) and Nelson and Schaefer (1983).

2.3 Cox, Ingersoll, and Ross (1985b) model

In this model, the term structure of interest rates at time t is given by the short-term interest rate r which follows the square root process

$$dr = \omega\left(\varphi - r\right) dt + \upsilon\sqrt{r}dz, \tag{10}$$

where ω, φ, and υ are positive constants, and z is a Wiener process. In this model, the interest r is pulled towards its long-term mean φ at the rate ω.

The price, $P(r, \tau)$, of any interest-rate contingent claim is the solution to the partial differential equation

$$\frac{1}{2}\upsilon^2(r)P_{rr} + \left(\omega(\varphi - r) - \lambda(r,t)\upsilon(r)\right)P_r + P_\tau - rP = 0, \tag{11}$$

subject to appropriate terminal and boundary conditions. Here, $\lambda(r, t)$ is the market price of short term interest rate risk, and is supposed to be $\lambda(r, t) = \lambda\sqrt{r}/\upsilon$, where λ is a constant. As in the previous two-factor models, there will be positive risk premiums when λ is negative.

For a discount bond maturing at time T, the terminal condition is $P(T, T) = 1$, and the solution of (11) gives

$$P(r, \tau) = A(\tau)e^{-B(\tau)r}, \tag{12}$$

where

$$\begin{aligned} A(\tau) &= \left(\frac{2\gamma e^{(\omega+\lambda+\gamma)\tau/2}}{(\gamma+\omega+\lambda)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right)^{\frac{2\omega\varphi}{\upsilon^2}}, \\ B(\tau) &= \frac{2\left(\exp\left(\gamma\tau\right)-1\right)}{(\gamma+\omega+\lambda)\left(e^{\gamma\tau}-1\right)+2\gamma}, \\ \gamma &= \sqrt{(\omega+\lambda)^2+2\upsilon^2}. \end{aligned}$$

3 Estimation and implementation of the models

3.1 The Data

To estimate the models from historical data, we use monthly one-month U.S. Treasury bill yields and one- and five-year taxable, noncallable, U.S. Treasury bonds yields for the period June of 1964 through December of 1989 (307 observations). The T-bill data set was originally constructed by Fama (1984) and subsequently updated by the Center for Research in Security Prices (CRSP). The yields are based on the average of bid and ask prices for Treasury bills and are normalized to reflect a standard month of 30.4 days. The Treasury bond yields are obtained from the Fama and Bliss (1987) data set, updated by CRSP as well. All yields are expressed in annualized form. This data set has been used by Chan et al. (1992), Longstaff and Schwartz (1992), Longstaff and Schwartz (1993), and Nowman (1997).

Table 1 shows the means, standard deviations, and first three autocorrelations of the one-month T-bill yield, the five-year Treasury bond yield, and the spread between one-month T-bill and five-year Treasury bond yields. The unconditional means of the one-month, five-year, and spread yields are 6.829%, 7.936%, and -1.107% , with standard deviations of 2.809%, 2.518%, and 1.463%, respectively[6]. The time series of the three rates during the sample period are shown in Figure 1.

3.2 Estimation of the Longstaff and Schwartz (1992) model

LS express the stochastic processes for r and V, given in expressions (1) and (2), as difference equations for r and V, that can be rearranged to give the following econometric

[6]Note that the statistics for the one-month rate are slightly different from those reported by Chan et al. (1992) and Longstaff and Schwartz (1992). The difference is due to the way of compounding the monthly rates. We use discretely compounded annual interest rates. Plots of the one-month rate and its actual volatility are practically identical to the ones shown by these authors.

model

$$\begin{aligned} r_{t+1} - r_t &= \alpha_0 + \alpha_1 r_t + \alpha_2 V_t + e_{t+1}, & (13) \\ e_{t+1} &\sim N(0, V_t), \\ V_t &= \beta_0 + \beta_1 r_t + \beta_2 V_{t-1} + \beta_3 e_t^2 & (14) \end{aligned}$$

This is a GARCH(1,1) model with conditional variance in the mean equation. In this model, the level of the interest rate appears as an additional regressor in both the mean and the variance equations.

The six stationary parameters of the LS model ($\alpha, \beta, \gamma, \delta, \eta$, and ξ) can be obtained from the estimated coefficients of the GARCH model. However the relationship between the parameters and the coefficients is very complex. Consequently, Longstaff and Schwartz (1993) suggest an alternative way of computing the parameters of their continuous-time model. They estimate the GARCH model to obtain a time series of conditional variances, that is used together with the original time series of interest rates to calculate $\alpha, \beta, \gamma, \delta, \eta$, and ξ as follows

$$\begin{aligned} \alpha &= \min\left(\frac{V_t}{r_t}\right), \\ \beta &= \max\left(\frac{V_t}{r_t}\right), \\ \delta &= \frac{\alpha(\alpha+\beta)\left(\beta \mathrm{E}\{r_t\} - \mathrm{E}\{V_t\}\right)}{2\left(\beta^2 \mathrm{Var}\{r_t\} - \mathrm{Var}\{V_t\}\right)}, \\ \gamma &= \frac{\delta(\beta \mathrm{E}\{r_t\} - \mathrm{E}\{V_t\})}{\alpha(\beta-\alpha)}, \\ \xi &= \frac{\beta(\alpha+\beta)(\mathrm{E}\{V_t\} - \alpha \mathrm{E}\{r_t\})}{2\left(\mathrm{Var}\{V_t\} - \alpha^2 \mathrm{Var}\{r_t\}\right)}, \\ \eta &= \frac{\xi(\mathrm{E}\{V_t\} - \alpha \mathrm{E}\{r_t\})}{\beta(\beta-\alpha)}, \end{aligned}$$

where $\mathrm{E}\{\cdot\}$ and $\mathrm{Var}\{\cdot\}$ denote expected value and variance, respectively.

In Table 2 we report the estimates of the GARCH model[7]. We see that all the coefficients are significant at the 0.01 level except the coefficient of the conditional variance in the mean equation (α_2).

In Figure 2 we plot the actual volatility [8] of one-month Treasury bills and the forecast volatility implied by the estimates of the GARCH model. We see that the volatility increases with the level of the interest rate, which is consistent with the CIR model.

Finally, we use our time series of interest rates and GARCH volatilities to calculate $\alpha, \beta, \gamma, \delta, \eta$, and ξ. The results are shown in Table 3. The conditional volatilities are also needed to price bonds, since the short-term volatility is one of the state variables in the LS model.

3.3 Estimation of the Schaefer and Schwartz (1984) and CIR (1985b) models

In these models, the dynamics of the state variables can be represented as

$$dx = a(b - x)dt + cx^d dz, \tag{15}$$

where the relationship of a, b, c and d with the original parameters of the processes is given in the following table

[7] We use the econometric package EViews 3.1 to estimate the model.

[8] Measured as the absolute value of the month-to-month change in the one-month Treasury bill yield.

	Spread	Long-term rate	Short-term rate
x	s	l	r
a	m	κ	ω
b	μ	θ	φ
c	γ	σ	υ
d	0	1/2	1/2

To estimate these models from historical data, we follow Nowman (1997) and use the Gaussian estimation method. We first express equation (15) in a discrete-time setting as

$$x_t = e^{-a\Delta t}x_{t-1} + \theta\left(1 - e^{-a\Delta t}\right) + \eta_t \qquad \text{(t=1, 2 \ldots T)} \tag{16}$$

where Δt is the time interval, and η_t satisfies the conditions

$$\begin{aligned} E\left(\eta_t\right) &= 0, \\ E\left(\eta_s \eta_t\right) &= 0, \qquad (s \neq t), \\ E\left(\eta_t^2\right) &= \frac{c^2 x_{t-1}^{2d}}{2a}\left(1 - e^{-2a\Delta t}\right). \end{aligned}$$

We then obtain the parameter estimates maximizing[9] the Gaussian log-likelihood function of the process (16), given by

$$L(a,b,c,d) = -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln E\left(\eta_t^2\right) - \frac{1}{2}\frac{\left[x_t - e^{-a\Delta t}x_{t-1} - b\left(1 - e^{-a\Delta t}\right)\right]^2}{E\left(\eta_t^2\right)}. \tag{17}$$

This expression is the exact log-likelihood function for the spread process, but only an approximation of it for the long and short rate processes (see Brown and Schaefer (1996)).

[9] We use the FORTRAN routine MINIM for function minimization using the simplex method.

In the SS model, the long-term interest rate process describes the dynamics of the yields of a riskless consol bond. Brennan and Schwartz (1982) approximate the consol rate, l, by the annualized yield to maturity on the highest-yielding U.S. Treasury Bond with a maturity exceeding 20 years; if no such bond is available in a particular month, they use the highest-yielding bond with a maturity of more than 15 years. In this paper, because of data constraints, and only as an approximation, five-year U.S. Treasury bond yields are used instead.

Table 4 shows the parameter estimates for the SS and CIR models. t-statistics are provided in parentheses. We see that the parameters are significant at the 0.01 level, with the exception of the mean reversion speed (κ) of the long rate process in the SS model, which is significant only at the 0.10 level[10]. The long-term means, θ, of the long rate, spread, and short rate processes are 8.6271%, -1.1210%, and 6.9965%, respectively, which are close to the unconditional means (7.936%, -1.107%, and 6.829%, respectively). The speeds of adjustment, κ, of each rate to its mean are 0.2083, 2.2942, and 0.6647, for the long rate, spread, and short rate, respectively. This implies a mean half-life[11] of 3.3278, 0.3021, and 1.0428 years, respectively, indicating that the mean reversion effect is stronger in the spread process. Finally, the estimates of the parameter σ are 0.0509, 0.0313, and 0.1140 for the long rate, spread, and short rate, respectively.

4 Pricing Bonds

Before pricing interest-rate claims with these three models we must estimate the corresponding market prices of interest-rate risk, λ. We do so from monthly cross-sections of one-month Treasury bill and five-year Treasury bond yields, using the parameter esti-

[10]This fact, however, does not affect the valuation of bonds, since the mean reversion parameters of the long rate process do not appear in the pricing formulas.

[11]Defined as the time that the rate needs to achieve the halfway between the current level and the long-run mean θ. It is computed as $\ln(2)/\kappa$.

mates obtained from the time-series data. Each month, we minimize the sum of squared errors (SSE), i.e.

$$\min_{\lambda} \sum_{i=1}^{2} (y_i - \hat{y}_i)^2 \tag{18}$$

where y_i, and $\hat{y}_i$, stand for market yield and theoretical yield, respectively, and $i = 1, 2$ represents the two maturities considered.

Figure 3 plots the estimates of each λ in each model. We see that the parameter is unstable through time, and ranges from -0.6191 to -0.3128 in the LS model, from -0.7245 to 0.1984 in the SS model, and from -0.4670 to 0.4710 in the CIR model. To compute bond prices, we use the sample means of λ, given by -0.4533 in the LS model, -0.3396 in the SS model, and -0.0687 in the CIR model. Note that these means imply positive risk premiums.

To price bonds with the SS model, we assume that $\hat{s} \approx \hat{\mu}$, even though the value of the spread is not equal to $\hat{\mu}$ in our sample[12]. This assumption is supported by the fact that bond prices are not very sensitive to the value of $\hat{s}$.

4.1 One-Month Treasury Bills

Figure 4 graphs one-month T-bill yields versus the predicted yields of the LS, SS, and CIR models (computed using equations (3), (9), and (12), respectively). The approximation on the three cases seems to be very good. To compare the models, we use the mean absolute percentage error (MAPE) measure, defined as

$$\text{MAPE} = \frac{1}{n} \sum_{i=1}^{n} \frac{|y_i - \hat{y}_i|}{y_i}, \tag{19}$$

[12]The mean value of the spread is -0.01107, while $\hat{\mu} = -0.0064$

where n is the number of observations.

Given the characteristics of the models, we expect two results. First, LS nest the single-factor CIR model within their two-factor model of the term structure. Thus, the LS model is likely to be more successful at describing the evolution of short rates than the CIR model, and should be superior in pricing contingent claims. Second, LS use the short-term interest rate and its volatility as state variables to describe the term structure of interest rates, while SS use the long-term rate and the spread between the short-term and the long-term rates. Given that, as LS argue, the equilibrium approach has several advantages over the arbitrage approach[13], the LS model should be superior also to the SS model in pricing short-term contingent claims.

The second column of Table 5 reports the MAPE for each model for one-month T-bills. We see that, as expected, the pricing error of the LS model is substantially smaller than the others (0.2042% versus 1.8063% for the SS model and 1.0701% for the CIR model). Nonetheless, the three models perform reasonably well. Note that the pricing error of the one-factor CIR model is smaller than that of the SS two-factor model, which is not surprising given that, in the CIR model, the volatility of the short-term rate depends on the level of the riskless rate, which is supported by the empirical evidence.

4.2 One-Year Treasury Bonds

To study the pricing of medium term bonds, we use 1-year Treasury bond data. Figure 5 plots observed one-year T-bond yields versus theoretical yields for the three models. Note that the pricing errors are greater than before. Table 5 shows that the MAPE of the LS model dramatically increases to 10.0973%, indicating that this model has problems to pick up the dynamics of one-year rates. The pricing errors in the CIR model also

[13] The variables which determines contingent claim prices, the dynamics of these variables, and the functional forms of the market prices of risk are all endogenously determined.

increase substantially, to 8.9728%, and the MAPE in the SS model increases the least, to 6.0622%, which is probably due to the fact that SS incorporate explicity information about diferent points of the yield curve.

4.3 Five-year Treasury Bonds

Finally, it is interesting to analyze the ability of the models to price longer term bonds. Since the SS model uses the long-term interest rate as one of the state variables, we expect it to be superior to the CIR model, that uses only the short-term rate, and to the LS model which does not use any long-term state variable. Moreover, the LS model should perform better than the CIR model since it is a general case of the CIR model that uses two short-term state variables. Figure 6 confirms our expectations. We see that the pricing error increases in the LS and the CIR models, while it decreases significantly in the SS model. In Table 5 we see that the LS and CIR models do a poor job pricing 5-year Treasury bonds, incurring in MAPEs of 14.6607% and 18.3520%, respectively. However, the SS model prices the five-year bond ever better that one-month bond, producing a MAPE of just 1.2137%.

4.4 Linear Yield Prediction

To further compare the performance of the models, we test them for unbiased linear prediction of the actual one-month, one-year, and five-year U.S. Treasury bond yields.

For each model, the following linear regression is estimated

$$\begin{aligned} \hat{y}_i &= \alpha_0 + \alpha_1 y_i + \varepsilon_i \qquad (20) \\ \varepsilon_i &\sim \mathrm{N}(0, \zeta), \end{aligned}$$

where ζ is constant.

If the estimated intercept term, α_0, is not significantly different from 0, and the coefficient of the actual bond yields, α_1, is not significantly different from 1, we cannot reject the hypothesis that the computed bond yield is a linear unbiased predictor of the actual bond yield. The R^2 of the regressions provides an additional measure of the goodness of fit of the models.

Table 6 provides the results of the regressions of the computed yields on the actual ones. For one-month Treasury bills (rows 2-7), the intercept term is significantly different from 0, and the slope is significantly different from 1 (at the .01 level) for the three models. Therefore, the hypothesis that the model is unbiased linear predictor is rejected in the three cases. However this should not be discouraging since the R^2 of the regressions are almost identically equal to 1, implying that the biases are very small.

In rows 8-13 we present the results for 1-year Treasury bonds. We reject, at the .01 level, that the LS, SS and CIR models are unbiased linear predictors of 1-year Treasury bond yields, since the intercept term is significantly different from 0 or the slope is significantly different from 1. These results should be interpreted with care since a model can be a biased linear predictor, but can explain much more of the variance of the actual yields than a model for which we fail to reject the null hypothesis.

Finally, in rows 14-19 we see that for 5-year Treasury bonds we reject, at the .01 level, that the CIR model is unbiased linear predictor of 5-year Treasury bond yields. Moreover, the goodness of fit of the model, as measured by R^2, is poor (0.73043). However, we fail to reject that LS and SS models are unbiased linear predictors of the five-year yields. This result is more interesting for the SS model, given that the R^2 is very high (0.99757). In general, these results are consistent with those given by the MAPE measure.

5 Summary and Conclusions

This paper compares the Longstaff and Schwartz (1992) general equilibrium model and the Schaefer and Schwartz (1984) arbitrage model of the term structure of interest rates. The two models differ in: (1) the considerations they use to derive the partial differential equation for pricing assets; (2) LS use two short-term state variables, while SS use one short-term and one long-term variable; and, (3) LS develop an exact solution, while SS offer an approximated analytical solution to the PDE.

Two results are expected. On the one hand, since the CIR model can be nested into the LS model, the LS model should perform better than the CIR model in pricing both short-term and long-term bonds. On the other hand, if the general equilibrium approach and the arbitrage approach are equivalent, the LS model should perform better (worse) than the SS model in pricing short-term (long-term) securities, since it uses two short-term state variables.

Our empirical results support both conjectures. For one-month Treasury bills, the three models perform relatively well, with mean absolute percentage errors (MAPE) smaller than 2%. The LS model performs best, with a MAPE of only 0.2042%. The performance of the three models decreases for medium-term bonds. For five-year bonds, however, the results are mixed but expected, since the mean relative errors are greater than 14% and 18% in the LS and CIR models, while that for the SS model the error is just about 1%.

Thus, we see that the three models considered have difficulties adjusting to the market yield curve. To fit the curve, generalized versions of these models can be used (in the ways suggested by Longstaff and Schwartz (1993) and Hull and White (1990a)). However, even in this case, there is no guarantee that the models will price interest-rate options accurately (see Navas (1999)).

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Table 1: Descriptive statistics.

	r	l	s
Mean	0.06829	0.07936	-0.01107
Standard Deviation	0.02809	0.02518	0.01463
ρ_1	0.94220	0.98437	0.82618
ρ_2	0.89481	0.96679	0.73261
ρ_3	0.85182	0.95188	0.63359

Summary statistics for one-month U.S. Treasury bill yield, r, five-year Treasury bond yield, l, and the spread between one-month U.S. Treasury bill and five-year U.S. Treasury bond yields, s, from June 1964 through December 1989. Monthly data is used (307 observations). The autocorrelation coefficient of order i is denoted as ρ_i.

Table 2: GARCH estimates of the Longstaff and Schwartz (1992) two-factor model.

	α_0	α_1	α_2	β_0	β_1	β_2	β_3	Log-Likelihood
Value	0.003982	-0.066643	3.906229	-0.000010	0.000289	0.693590	0.190713	1090.93
z-statistic	3.24	-2.43	0.35	-3.47	3.49	10.34	3.55	
p-value	0.0012	0.0149	0.7262	0.0005	0.0005	0.0000	0.0004	

The continuous-time Longstaff and Schwartz (1992) model is approximated by the following GARCH(1,1) model with conditional variance in the mean equation:

$$\begin{aligned} r_{t+1} - r_t &= \alpha_0 + \alpha_1 r_t + \alpha_2 V_t + e_{t+1}, \\ e_{t+1} &\sim N(0, V_t), \\ V_t &= \beta_0 + \beta_1 r_t + \beta_2 V_{t-1} + \beta_3 e_t^2. \end{aligned}$$

The parameters are obtained from the maximum likelihood estimates of the GARCH model. The data are monthly one-month U.S. Treasury bill yields during the period from June 1964 to December 1989 (306 observations).

Table 3: Stationary parameters of the Longstaff and Schwartz (1992) two factor model.

Parameter	MLE Estimate
α	0.000102
β	0.01248
γ	2.7553
δ	0.00452
η	0.2258
ξ	0.4421

The parameters are obtained from the maximum likelihood estimates of the parameters of a GARCH model for monthly changes in the one-month U.S. Treasury bill yield. The data are monthly, from June 1964 through December 1989 (306 observations).

Table 4: Gaussian estimates of the Schaefer and Schwartz (1984) (SS) and the Cox, Ingersoll and Ross (1985b) (CIR) two-factor models.

		a	b	c	d	Log-Likelihood
SS	Long-term process	0.208287	0.086271	0.050864	0.5	1255.7598
		(1.6836)	(5.6389)	(24.3462)		
	Spread process	2.294191	-0.011210	0.031346	0.0	1033.8872
		(4.9093)	(-4.1581)	(22.6858)		
CIR	Short-rate process	0.664686	0.069965	0.114051	0.5	1041.2073
		(2.7103)	(7.7110)	(23.9853)		

The data are monthly one-month U.S. Treasury bill yields and five-year U.S. Treasury bond yields, during the period from June 1964 to December 1989 (306 observations). It is assumed that, in the Shaefer and Schwartz model, the long-term interest rate follows a square root process. The continuous-time model is $dx = a\,(b - x)\,dt + cx^d dz$, where $x = l$ and $x = s$ for the long-term and spread processes, respectively, in the Shaefer and Schwartz model, and $x = r$ in the CIR model. Gaussian estimates (expressed in yearly basis) with t-statistics in parentheses are presented for each model.

Table 5: Mean Absolute Percentage Errors (MAPE) for Treasury bills and Treasury bonds of different maturities.

Model	Maturity		
	1 Month	1 Year	5 Year
Longstaff-Schwartz	0.2042%	10.0973%	14.6607%
Schaefer-Schwartz	1.8063%	6.0622%	1.2137%
CIR	1.0701%	8.9728%	18.3520%

Comparison of Longstaff and Schwartz (1992) two-factor general equilibrium model, Schaefer and Schwartz (1984) two-factor model, and Cox Ingersoll and Ross (1985) one-factor model. The data are monthly one-month U.S. Treasury bill and five-year U.S. Treasury bond yields from June 1964 to December 1989. The comparisons are made in terms of the mean absolute percentage error (MAPE) defined as $\frac{1}{n}\sum_{i=1}^{n}\frac{|y_i-\hat{y}_i|}{y_i}$, where y_i is the actual bond yield in month i, and $\hat{y}_i$ is the computed bond yield in month i.

Table 6: Test for unbiased linear prediction of market yields.

Maturity	Model	α_0	α_1	R^2
1 month	Longstaff-Schwartz	0.0001*	0.9999**	1.0
		(175.07)	(-5.64)	
	Schaefer-Schwartz	0.0019*	0.9792**	0.99823
		(11.03)	(-8.67)	
	CIR	0.0019*	0.9755**	1.0
		(45020.2)	(-42628.7)	
1 year	Longstaff-Schwartz	−0.0074*	1.0224	0.91259
		(-5.12)	(1.23)	
	Schaefer-Schwartz	0.0005	0.9551**	0.96897
		(0.59)	(-4.59)	
	CIR	0.0125*	0.7697**	0.91241
		(11.49)	(-16.87)	
5 years	Longstaff-Schwartz	0.0005	0.9399	0.72986
		(0.19)	(-1.83)	
	Schaefer-Schwartz	-0.0002	1.0018	0.99757
		(-0.91)	(0.63)	
	CIR	0.0504*	0.2998**	0.73043
		(58.08)	(-67.14)	

Comparison of Longstaff and Schwartz (1992) two-factor general equilibrium model, Schaefer and Schwartz (1984) two-factor model, and Cox Ingersoll and Ross (1985) one-factor model. Each model is tested for unbiased linear predictor of the actual one-month U.S. Treasury bill and five-year U.S. Treasury bond yields from June 1964 through December 1989. For each model, the following regression is estimated:

$$\begin{aligned} \hat{y}_i &= \alpha_0 + \alpha_1 y_i + \varepsilon_i \\ \varepsilon_i &\sim N(0, \zeta) \end{aligned}$$

where y_i is the actual bond yield in month i, $\hat{y}_i$ is the computed bond yield in month i, and ζ is a constant. The t-values are in parentheses in columns 3 and 4, respectively.

* Significantly different from 0 at the .01 level
** Significantly different from 1 at the .01 level

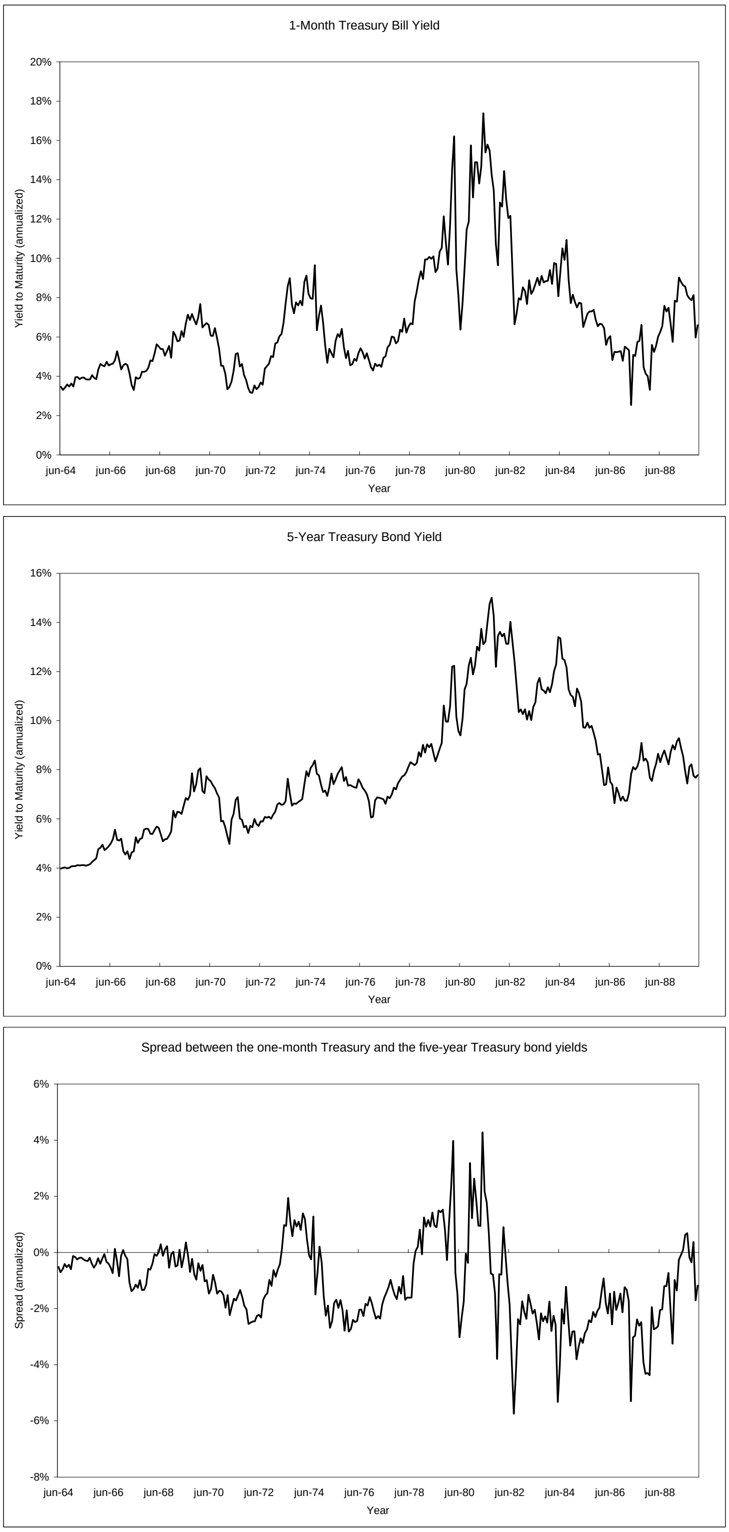


Figure 1: Short-term rate, long-term rate, and spread from June 1964 through December 1989.

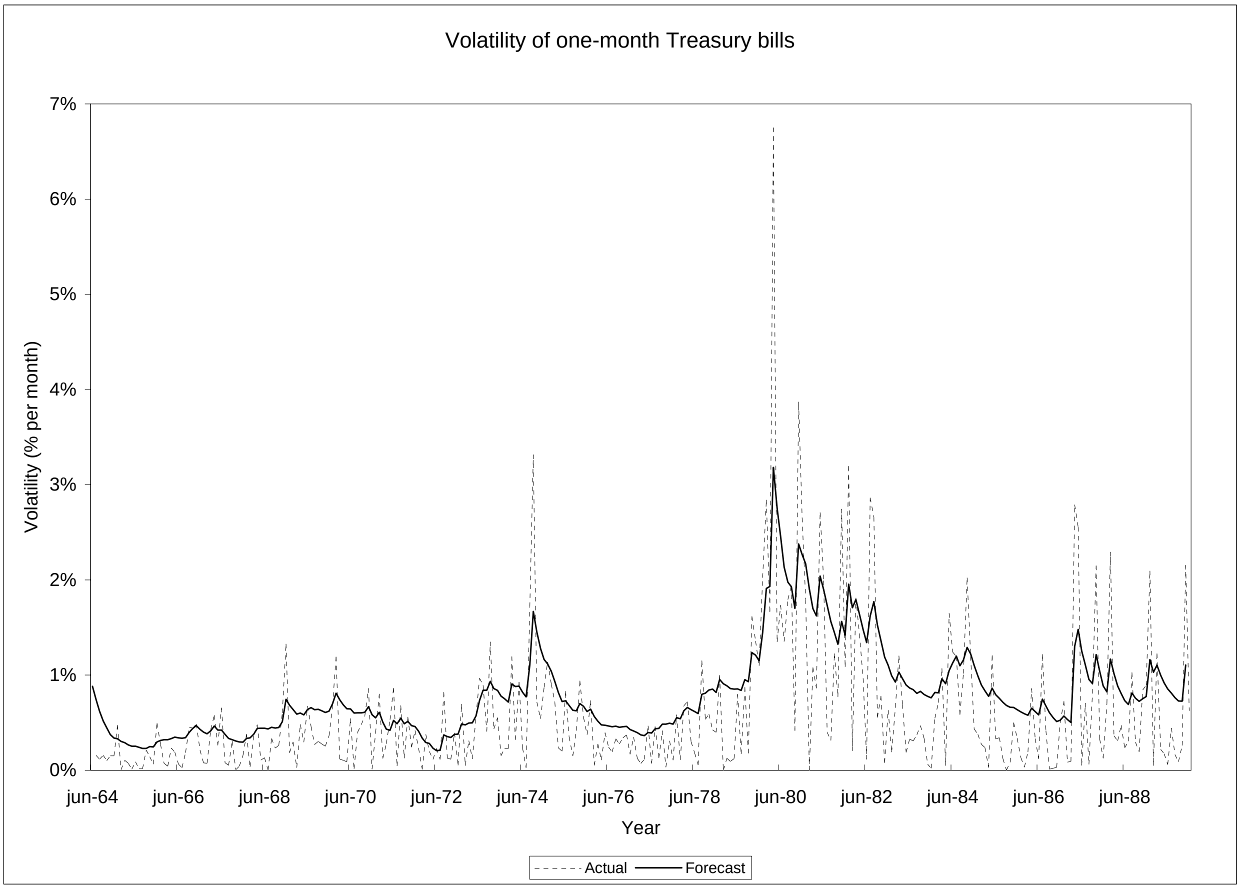


Figure 2: Volatility of one-month Treasury bills.

The actual volatility (dashed line) is measured as the absolute value of the month-to-month change in the one-month Treasury bill yield. The forecast (solid line) is the square root of the conditional variance implied by the estimates of the GARCH model.

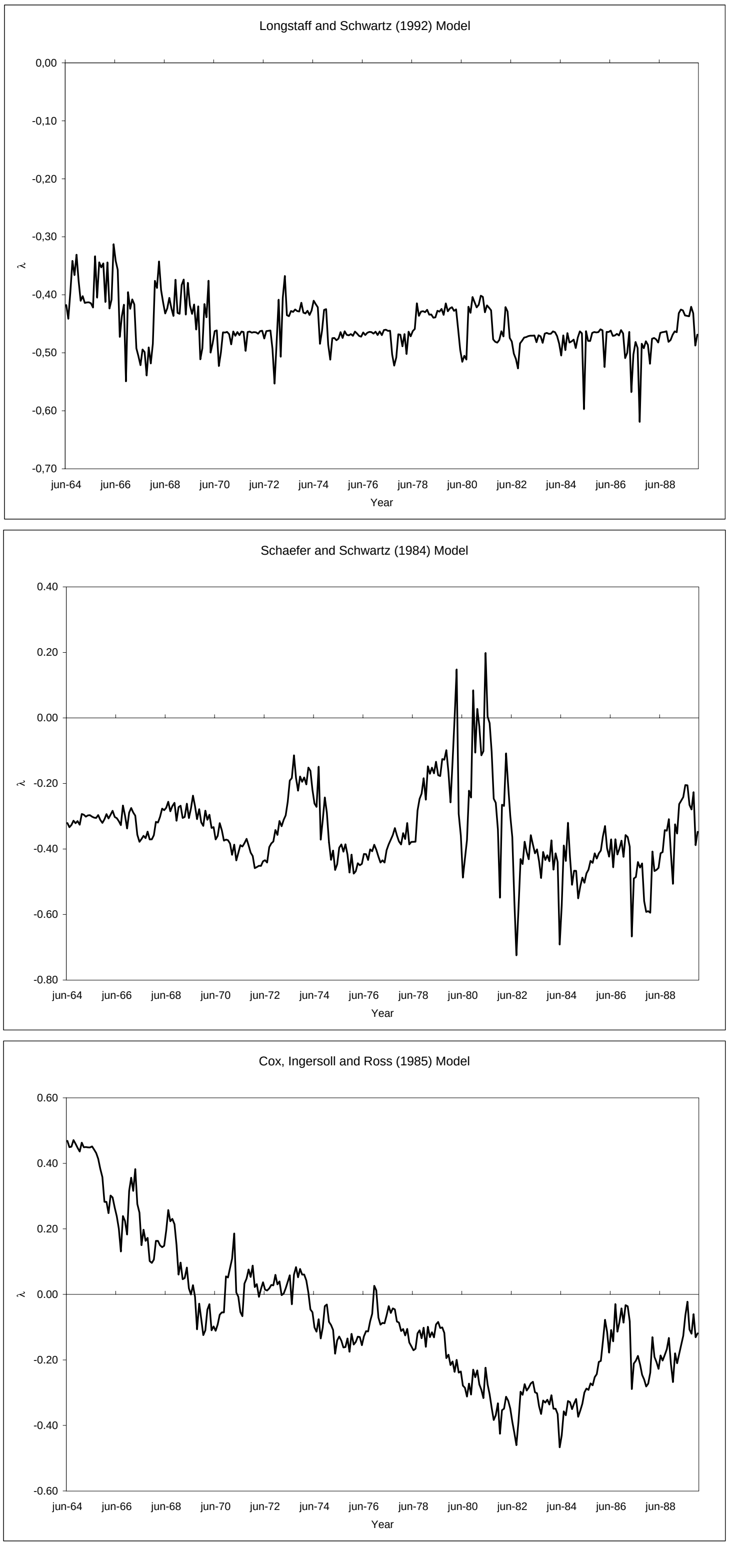


Figure 3: Estimation of the market price of risk (λ).

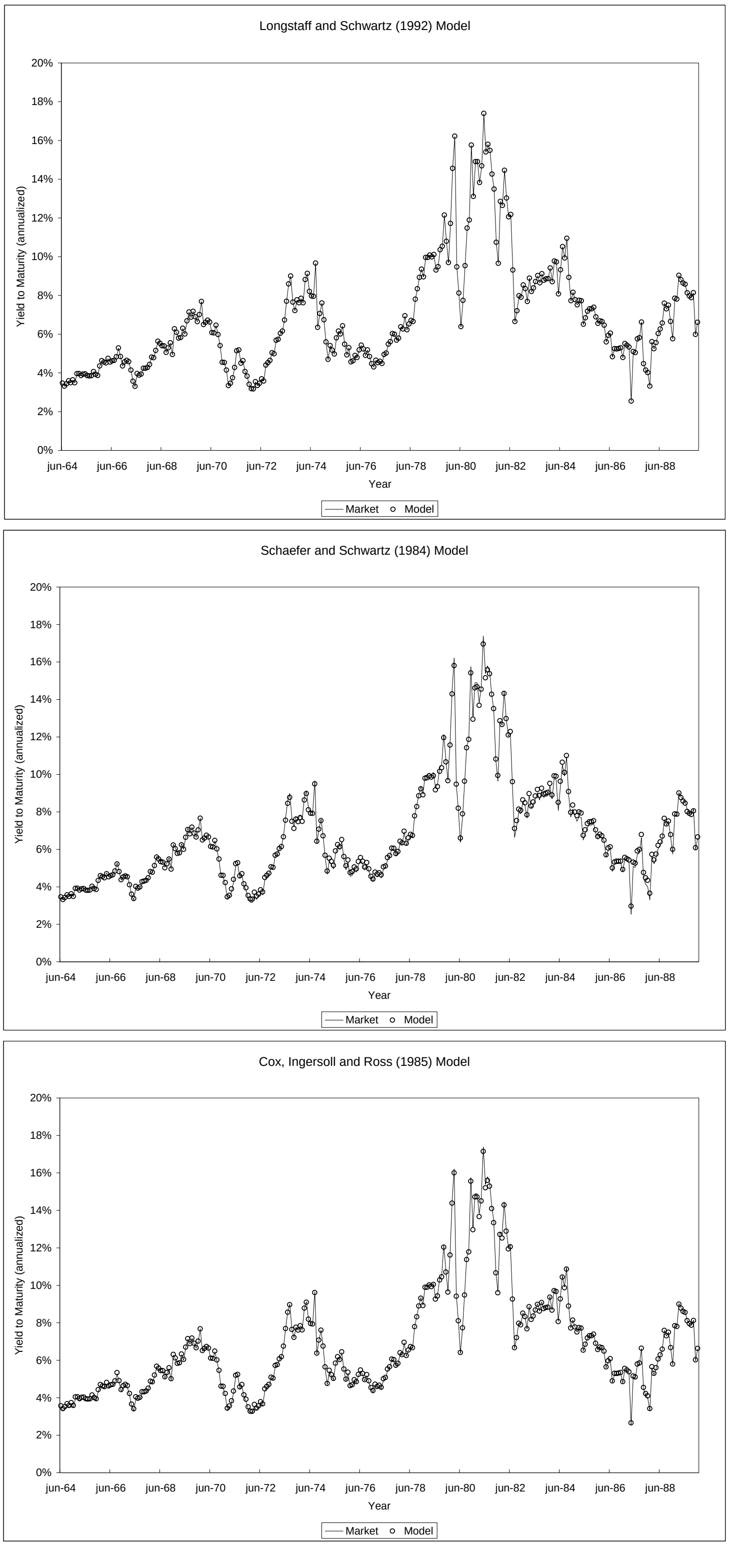


Figure 4: One-month Treasury bill yields.

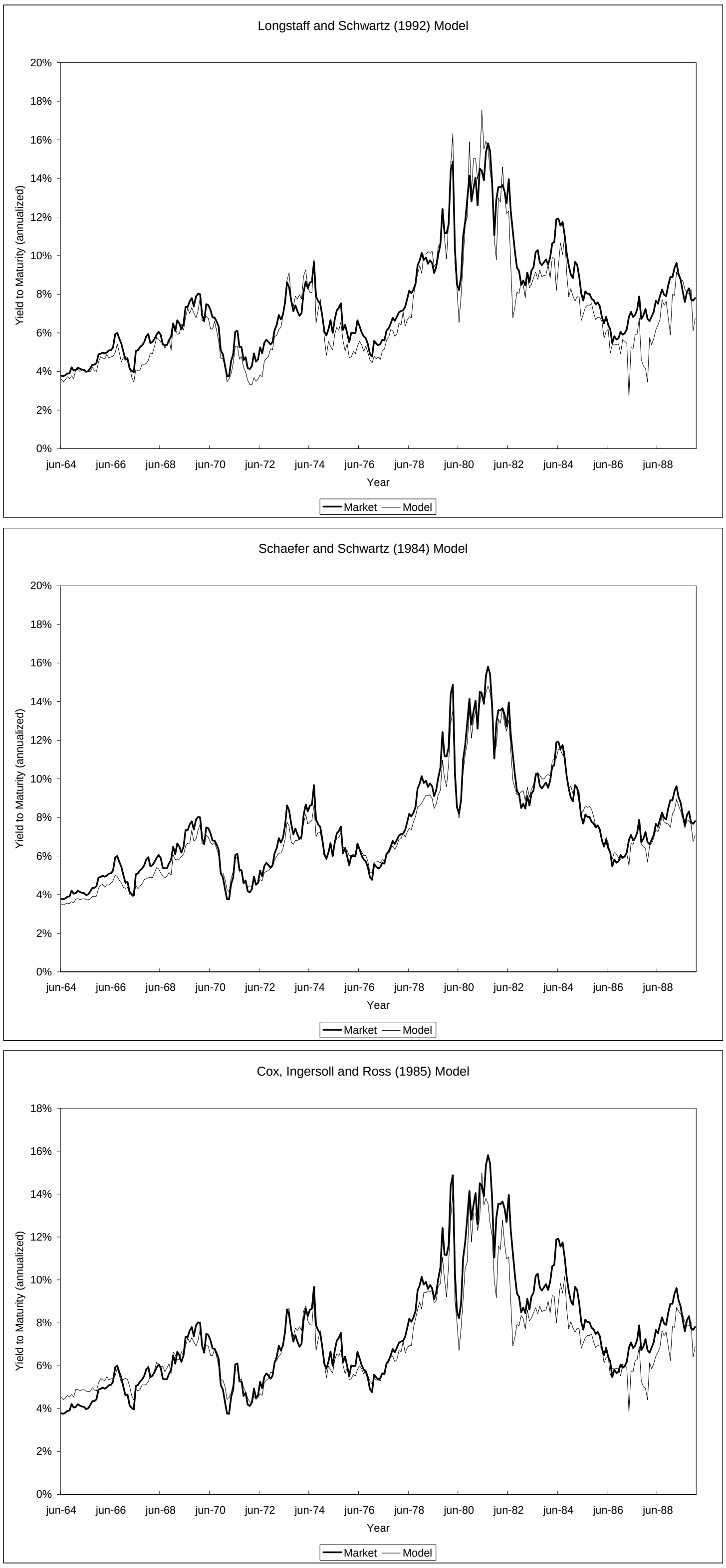
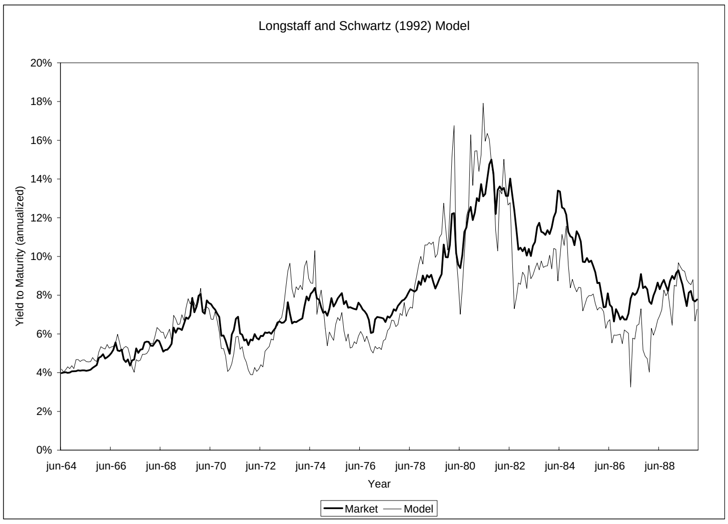
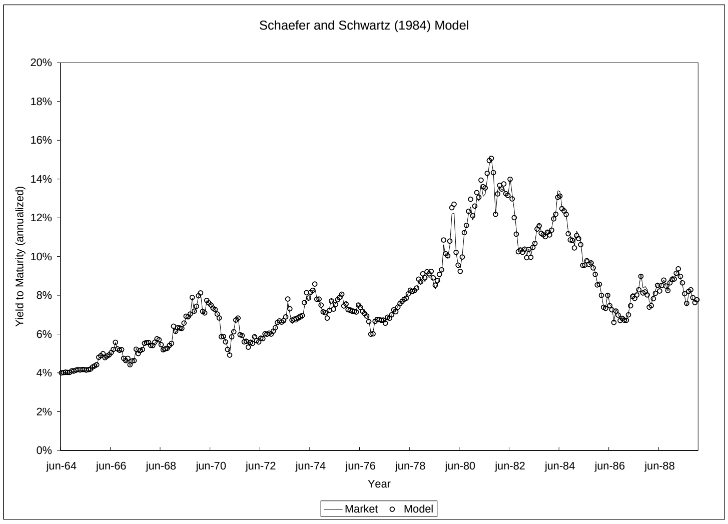


Figure 5: One-year Treasury bond yields.





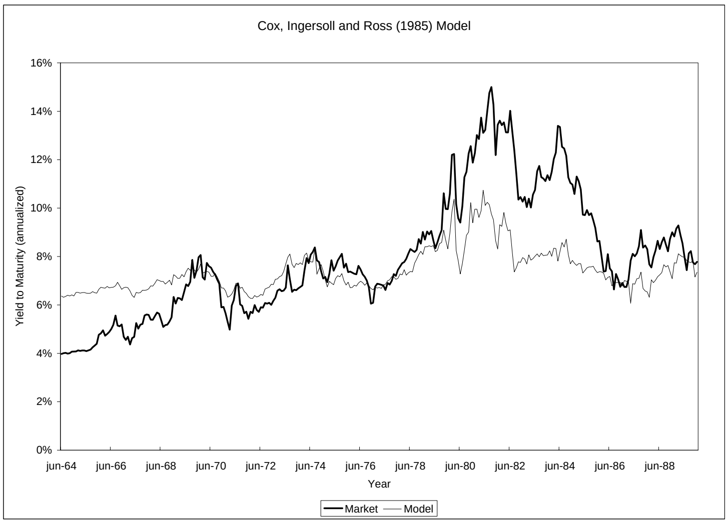


Figure 6: Five-year Treasury bond yields.