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# ON THE ROBUSTNESS OF LEAST-SQUARES MONTE CARLO (LSM) FOR PRICING AMERICAN DERIVATIVES

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**Abstract**

This paper analyses the robustness of Least-Squares Monte Carlo, a technique recently proposed by Longstaff and Schwartz (2001) for pricing American options. This method is based on least-squares regressions in which the explanatory variables are certain polynomial functions. We analyze the impact of different basis functions on option prices. Numerical results for American put options provide evidence that this approach is robust to the choice of polynomials. However, contrary to the claim of Longstaff and Schwartz (2001), it is not clear how many basis functions are required. For other derivatives, option prices are more affected by the type and number of basis functions used.

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Least-Squares Monte Carlo, option pricing, American options

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# On the Robustness of Least-Squares Monte Carlo (LSM) for Pricing American Derivatives

## Abstract

This paper analyses the robustness of Least-Squares Monte Carlo, a technique recently proposed by Longstaff and Schwartz (2001) for pricing American options. This method is based on least-squares regressions in which the explanatory variables are certain polynomial functions. We analyze the impact of different basis functions on option prices. Numerical results for American put options provide evidence that this approach is robust to the choice of polynomials. However, contrary to the claim of Longstaff and Schwartz (2001), it is not clear how many basis functions are required. For other derivatives, option prices are more affected by the type and number of basis functions used.

# 1 Introduction

How much do you pay for a certain asset if you know its final pay-off but you ignore when you will receive it? That is one of the main questions that academics and practitioners interested in American derivatives try to answer. The difficulty for answering this question is that we do not know when we will receive the reward promised by the asset. Thus, there exists a possibility of early exercise. At each exercise time before maturity, the optionholder must decide whether to exercise or to wait. This decision depends on the comparison, at each date, between the (known) immediate exercise value and the (unknown) continuation value.

Closed-form expressions for derivative prices exist in few cases. One example is an European option written on a stock whose price was derived by Black and Scholes (1973) and Merton (1973). Analytical expressions for the price of American options have been found, but numerical methods such as trees, finite difference schemes, quadrature routines or Monte Carlo simulation are usually required.

The Monte Carlo approach simulates paths for asset prices. An estimation of the option price is obtained by the discounted average of the option pay-offs computed for each path. This technique is appropriate to price options with complex features (path-dependence, multiple stochastic processes, random volatility, jumps, ...).

This technique is well suited for pricing European options, but it has not been widely applied to American derivatives. Recently, Longstaff and Schwartz (2001) have developed an algorithm to estimate the continuation value by a least-squares regression. This technique is known as Least-Squares Monte Carlo (LSM). These authors regress the discounted future cash-flows against a set of (basis) functions in the underlying asset prices. They claim that the type and number of basis functions have little effect on option prices.

This paper analyzes the robustness of the LSM approach for pricing American derivatives. For a put option, we find that LSM is indeed robust to the choice of the type, but not the number, of basis functions. For more complex derivatives, we find that the number and the type of functions do affect option prices.

This article is organized as follows. Section 2 reviews numerical methods for pricing American-style options. In Section 3, we briefly present the LSM technique and we provide a numerical example. Section 4 describes the set of basis functions used in this paper and we study the pricing of some American derivatives. Finally, Section 5 concludes the paper.

# 2 Numerical Methods for Pricing American Derivatives

Analytical solutions for the case of an American call option with discrete dividends have been derived by Roll (1977), Geske (1979) and Whaley (1981). The solution for the infinite horizon case is provided by McKean (1965). Recently, Ait-Sahlia (1996) and Ait-Sahlia and Lai (1996, 2000) have obtained closed-form expressions for the optimal exercise boundary.

Other analytical solutions have been obtained by the method of lines (see Rektorys (1982)). Carr and Faguet (1996) discretize the time derivative in the Black-Scholes PDE and then solve analytically the resulting sequence of ordinary differential equations. In a similar way, Carr (1998) also discretizes the time derivative and randomizes the expiration date of the American option.

Other authors obtain closed-form solutions for approximations to the original pricing problem. See Johnson (1983), Geske and Johnson (1984), Barone-Adesi and Whaley (1987), Bunch and Johnson (1992), Broadie and Detemple (1996), Ho *et al.* (1997), and Ju and Zhong (1999), among others. Ait-Sahlia and Carr (1997) and Ju (1998) compare some of these techniques.

Johnson (1983) presents an interpolation method based on regressing option prices against lower and upper bounds. A similar technique can be found in Broadie and Detemple (1996), who describe the lower bound (LBA) and the average of lower and upper bound (LUBA) methods.

Geske and Johnson (1984) apply the Richardson extrapolation technique[1] to price compound options. Several modifications by Bunch and Johnson (1992) and Ho*et al.* (1994, 1997) have been suggested.

In general, to price complex options, numerical techniques are required. Lattice methods are based on the discretization of the risk-neutral processes followed by the relevant variables. Then, backward induction in time is used to find the option price.

The binomial model was introduced by Cox *et al.* (1979) and Rendleman and Bartter (1979) and is based on the random walk approximation to the Brownian motion. Generalizations of this model have been suggested by Breen (1991), who proposes the "accelerated binomial method" with Richardson extrapolation to reduce the number of steps, and by Broadie and Detemple (1996). In some cases, trinomial trees, originally proposed by Parkinson (1977) and Boyle (1988), are used to increase the accuracy.

An alternative technique is the "finite difference" method. After building a grid of mesh points, an approximate solution of the PDE for the option price is obtained by replacing the partial derivatives with finite differences. Depending on how these differences are computed, we obtain the fully explicit[2], fully implicit or Crank-Nicolson method, respectively.[3]

For two or three dimensions, LOD (Locally One Dimensional) and ADI (Al-

[1] This technique has also been used to accelerate valuation methods by Breen (1991), Huang *et al.* (1996), Carr and Faguet (1996), Carr (1998) and Ju (1998).

[2] It can be seen that the explicit method is equivalent to approximate the diffusion process by a discrete trinomial tree. See Clewlow and Strickland (1998) for details.

[3] The first two methods were introduced by Schwartz (1977) and Brennan and Schwartz (1977, 1978) while the Crank-Nicolson method was first used in option pricing by Courtadon (1982).

ternating Direction Implicit) methods are developed.[4] For higher dimensions, Monte Carlo simulation is required.

American options can also be priced without approximating the stochastic process for the underlying asset or the partial differential equation for the option price. This is the case of quadrature techniques, which are based on approximating the integral that gives the option price. Examples of this technique are the trapezoidal and Simpson's rules.

Some authors use the integral representation method. Kim (1990), Jacka (1991), and Carr *et al.* (1992) decompose the price of an American put into the price of an European put option plus the early exercise premium, that is expressed as an integral.

Different approximations of this integral have been proposed. Huang *et al.* (1996) approximate the integrand with step functions to decrease the number of early exercise points. Ju (1998) recognizes that the premium does not depend critically on the early exercise boundary and approximates this boundary with a multipiece exponential function. Numerical results show that this approximation together with the method in Broadie and Detemple (1996) and the randomization technique by Carr (1998) are the most accurate methods for pricing American options.

Recently, Bunch and Johnson (2000) have derived exact expressions for the critical stock price function and the American put price in the perpetual and finite cases. Finally, Ait-Sahlia and Lai (2000) propose two different solutions based on a piecewise linear approximation of the early exercise boundary.

Monte Carlo simulation was introduced in finance by Boyle (1977). For a recent survey see Boyle *et al.* (1997). As shown by Harrison and Kreps (1979) and Harrison and Pliska (1981), the value of an option is the risk-neutral expectation of its discounted future value. This expectation is estimated by computing the

[4]See Morton and Mayers (1998) for details.

average of a large number of pay-offs.

Monte Carlo simulation is suitable for path-dependent options and can be extended to price options with multiple stochastic processes, random volatility, jumps,.... Its major disadvantage is that it is computationally intensive and inefficient. To mitigate this problem, variance reduction techniques, such as antithetic variables and control variates, have been developed.

Tilley (1993) prices American options using this technique. At each exercise date, he orders simulated paths by asset prices and bundles them into groups. For each group, an optimal exercise decision is taken.

Barraquand and Martineau (1995) reduce the dimensionality of the valuation problem grouping the simulated values into "bins". The transition probabilities between bins is determined by simulation and each bin is used as a decision unit to price the option.

Broadie and Glasserman (1997a) obtain point estimates and error bounds for American option prices. After showing that there are not unbiased estimates of these prices, they develop two (biased) estimates that converge asymptotically to the true price.

Broadie *et al.* (1997) and Raymar and Zwecher (1997) price American options on the maximum of several assets improving the techniques presented by Broadie and Glasserman (1997a) and Barraquand and Martineau (1995), respectively.

Ibañez and Zapatero (1998) suggest a general Monte Carlo simulation method for computing the optimal exercise boundary as the fixed point of an algorithm. To obtain this boundary, all the parameters but one are fixed.

Finally, non-parametric methods can also be used to price American options. This is the case of neural networks that tries to recover an unknown pricing function given historical data. Once the network has "learned" from the data, it is applied to out of sample data to determine the unknown price. See, for example, Hutchinson *et al.* (1994).

## 3 The Least-Squares Monte Carlo Approach

The main problem for pricing American options is the existence of several exercise dates. At each exercise time, the optionholder must decide whether to exercise the option or to wait. This decision depends on the comparison between the immediate exercise and the continuation values of the option.

Therefore, the optimal exercise decision relies on the estimation of the continuation value. Longstaff and Schwartz (2001) estimate it by a least-squares regression jointly with the cross-sectional information provided by Monte Carlo simulation. In this regression, they use a set of basis functions in the underlying asset prices. The fitted values are taken as the expected continuation values. Comparing these estimations with the immediate exercise values, they identify the optimal exercise decision. This procedure is repeated recursively going back in time. Discounting the obtained cash-flows to time zero, the price of the American option is found.

More formally, they assume a finite time horizon, $[0, T]$, in which they define a probability space,[5] $(\Omega, \mathbb{F}, P)$, and an equivalent martingale measure, $Q$. Let $C(\omega, s; t, T)$, $\omega \in \Omega$, $s \in (t, T]$ denote the path of option cash-flows, conditional on (a) the option being exercised after $t$ and (b) the optionholder following the optimal stopping strategy at every time after $t$.

The American option is approximated by its Bermuda counterpart, assuming a finite number of exercise dates $0 < t_1 < t_2 < \ldots < t_K = T$. The continuation value is, under no-arbitrage conditions, the risk-neutral expectation of the future discounted cash flows $C(\omega, s; t_i, T)$:

$$F(\omega; t_i) = E_Q\left[\sum_{j=i+1}^{K} \exp\left(-\int_{t_i}^{t_j} r(\omega, s)\, ds\right)\, C(\omega, t_j; t_i, T) \mid \mathbb{F}_{t_i}\right], \qquad (1)$$

where $r(\omega, s)$ is the risk-free rate and $\mathbb{F}_{t_i}$ is the information set at time $t_i$.

[5] This is a triple consisting of $\Omega$, the set of all possible sample paths $(\omega)$, $\mathbb{F}$, the sigma-algebra of events at time $T$ and $P$, a probability measure defined on the elements of $\mathbb{F}$.

The idea underlying the LSM algorithm is that this conditional expectation can be approximated by a least-squares regression for each exercise date. At time $t_{K-1}$, it is assumed that $F(\omega; t_{K-1})$ can be expressed as a linear combination of orthonormal basis functions $(p_j(X))$ such as Laguerre, Hermite, Legendre or Jacobi polynomials. That is

$$F(\omega; t_{K-1}) = \sum_{j=0}^{\infty} a_j p_j(X), \; a_j \in \mathbb{R}$$

which is approximated by

$$F_M(\omega; t_{K-1}) = \sum_{j=0}^{M} a_j p_j(X), \; a_j \in \mathbb{R}.$$

This procedure is repeated going back in time until the first exercise date.

## 3.1 A Numerical Example

To provide some intuition, Longstaff and Schwartz (2001) present a numerical example. Here we include a different one that shows that, if we use the LSM approach with few paths, an American option can have a lower price than its European counterpart.

We price an American put option on a non-dividend stock. The strike price, $E$, is 1.1 and there are three possible exercise dates. The continuously compounded risk-free interest rate is 0.05 and the volatility of the stock return, $\sigma$, is 0.2.

We simulate[6] eight paths of the underlying stock price as shown in the following table[7]

[6]Given that we are using a geometric Brownian motion, the simulation is exact and is sufficient to simulate prices at exercise dates. For other stochastic processes, the time step should be smaller.

[7]The symbol '*' denotes the in-the-money paths. Focusing on this type of paths improves the efficiency of the LSM method.

| Path | t = 0 | t = 1 | t = 2 | t = 3 | Pay-off at t = 3 |
|---|---|---|---|---|---|
| 1 | 1 | 0.917938 * | 1.272171 | 1.417021 | 0 |
| 2 | 1 | 1.133931 | 1.290983 | 1.669802 | 0 |
| 3 | 1 | 1.162833 | 0.917742 * | 1.228432 | 0 |
| 4 | 1 | 1.096706 * | 1.081163 * | 1.118280 | 0 |
| 5 | 1 | 1.056690 * | 0.871784 * | 0.818722 * | 0.281278 |
| 6 | 1 | 1.416442 | 1.672474 | 1.263264 | 0 |
| 7 | 1 | 0.937138 * | 0.945920 * | 0.861259 * | 0.238741 |
| 8 | 1 | 0.872576 * | 0.658605 * | 0.475270 * | 0.624730 |

The last column of this table shows the final pay-offs of an European option. Discounting these values at time zero and averaging them, the price of this option is 0.123162.

For an American option, the LSM approach maximizes its value at each exercise date along in-the-money (ITM) paths. For each date, $X$ denotes the underlying stock price and $Y$ represents the discounted cash-flows received at future dates if the option is not exercised.

At time two, there are five ITM paths (all but the first, the second and the sixth ones) and the values of $X$ and $Y$ are as follows

| Path | $Y$ | $X$ |
|---|---|---|
| 1 | — | — |
| 2 | — | — |
| 3 | $e^{-0.05} \times 0$ | 0.917742 |
| 4 | $e^{-0.05} \times 0$ | 1.081163 |
| 5 | $e^{-0.05} \times 0.281278$ | 0.871784 |
| 6 | — | — |
| 7 | $e^{-0.05} \times 0.238741$ | 0.945920 |
| 8 | $e^{-0.05} \times 0.624730$ | 0.658605 |

To decide whether to exercise the option, we must estimate the continuation value and compare it with the immediate exercise value, $1.1-X$. The continuation value is estimated by regressing $Y$ on a constant, $X$ and $X^2$, which gives

$$E[Y \mid X] = 2.848474 - 4.6539\ X + 1.871826\ X^2.$$

The exercise decision is shown in the following table

| Path | $1.1 - X$ | $E[Y \mid X]$ | Decision |
|---|---|---|---|
| 1 | — | — | — |
| 2 | — | — | — |
| 3 | 0.182258 | 0.1539056 | Exercise |
| 4 | 0.018837 | 0.0048106 | Exercise |
| 5 | 0.228216 | 0.2138467 | Exercise |
| 6 | — | — | — |
| 7 | 0.154080 | 0.1210645 | Exercise |
| 8 | 0.441395 | 0.5952915 | Wait |

We exercise the option in all the ITM paths except the eighth one, in which $1.1 - X < E[Y \mid X]$. Therefore, assuming that the option is not exercised before time two, the cash-flows to the optionholder are the following

| Path | t = 1 | t = 2 | t = 3 |
|---|---|---|---|
| 1 | — | 0 | 0 |
| 2 | — | 0 | 0 |
| 3 | — | 0.182258 | 0 |
| 4 | — | 0.018837 | 0 |
| 5 | — | 0.228216 | 0 |
| 6 | — | 0 | 0 |
| 7 | — | 0.154080 | 0 |
| 8 | — | 0 | 0.62473 |

We repeat this procedure at time one, when we also have five ITM paths. To compute the variable $Y$, we use the cash-flows to be received at time two or three (but not in both dates) for each path. The values of $X$ and $Y$ are as shown next

| Path | $Y$ | $X$ |
|---|---|---|
| 1 | $e^{-0.05} \times 0$ | 0.917938 |
| 2 | — | — |
| 3 | — | — |
| 4 | $e^{-0.05} \times 0.018837$ | 1.096706 |
| 5 | $e^{-0.05} \times 0.228216$ | 1.056690 |
| 6 | — | — |
| 7 | $e^{-0.05} \times 0.154080$ | 0.937138 |
| 8 | $(e^{-0.05})^2 \times 0.624730$ | 0.872576 |

Estimating again $Y$ on a constant and the first two powers of $X$, we obtain

$$E[Y \mid X] = 23.905695 - 47.1482\ X + 23.23217\ X^2,$$

which leads us to the following exercise decision

| Path | $1.1 - X$ | $E[Y \mid X]$ | Decision |
|---|---|---|---|
| 1 | 0.182062 | 0.202191 | Wait |
| 2 | — | — | — |
| 3 | — | — | — |
| 4 | 0.003294 | 0.1407488 | Wait |
| 5 | 0.043310 | 0.0255102 | Exercise |
| 6 | — | — | — |
| 7 | 0.162862 | 0.1244155 | Exercise |
| 8 | 0.227424 | 0.4539830 | Wait |

Consequently, the cash-flows of this American option at the three exercise dates are the following

| Path | t = 1 | t = 2 | t = 3 |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 |
| 3 | 0 | 0.182258 | 0 |
| 4 | 0 | 0.018837 | 0 |
| 5 | 0.043310 | 0 | 0 |
| 6 | 0 | 0 | 0 |
| 7 | 0.162862 | 0 | 0 |
| 8 | 0 | 0 | 0.62473 |

Thus, at time one, we exercise the option in the fifth and seventh paths. At time two, we exercise the option in the third and the fourth paths and, at time three, a non-zero cash-flow is received in the eighth path.

All the cash-flows in the second and the sixth paths are null because they are out-of-the money paths. For the first path, the cash-flows are also zero even though, at time one, the option is ITM. This can be explained because the optimal decision at this time was to wait.

Finally, discounting these cash-flows to the initial date and averaging them over all paths, the price for the American option is 0.114473, a 7% smaller than the European counterpart. Of course, this is a consequence of the reduced number of simulated paths. Increasing the number of paths leads to American option prices that are larger than European ones.

# 4 Numerical Results on the Robustness of LSM

It seems interesting to analyze what happens to the option price when we change the number or type of basis functions. In this paper, we use the following polynomials with up to ten terms.[8]

| Name | $f_n(x)$ |
|---|---|
| Powers | $W_n(x)$ |
| Legendre | $P_n(x)$ |
| Laguerre | $L_n(x)$ |
| Hermite A | $H_n(x)$ |
| Hermite B | $H_{e_n}(x)$ |
| Chebyshev 1st kind A | $T_n(x)$ |
| Chebyshev 1st kind B | $C_n(x)$ |
| Chebyshev 1st kind C | $T_n^*(x)$ |
| Chebyshev 2nd kind A | $U_n(x)$ |
| Chebyshev 2nd kind B | $S_n(x)$ |

where $n \geq 0$ denotes the degree of the polynomial.

These polynomials can be expressed in three alternative ways:

1. Explicit expression

$$f_n(x) = d_n \sum_{m=0}^{N} c_m g_m(x),$$

where $N$ takes different values for different polynomials as shown in Table 1.

2. Rodrigues' formula

$$f_n(x) = \frac{1}{a_n\ g(x)} \frac{\partial^n}{\partial x^n} \left[\rho(x)\ (g(x))^n\right]$$

3. Recurrence law

$$a_{n+1}\ f_{n+1}(x) = (a_n + b_n\ x)\ f_n(x) - a_{n-1}\ f_{n-1}(x)$$

[8]See Demidowitsch *et al.* (1980) for details on $T_n^*(x)$ and Abramowitz and Stegun (1972) for the remaining ones.

The coefficients and functions included in these expressions are shown in Tables 1, 2, and 3, respectively.

[ Insert Tables 1, 2, and 3 about here ]

From a theoretical point of view, it would be desirable to use an orthonormal basis of functions on which to project continuation values. This means that

$$\int_a^b f_n(x)\, f_m(x)\, dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

The values for the limits of this integral vary with the polynomials. See Abramowitz and Stegun (1972) for details. In most cases, the range of underlying prices ($X$) is different from the interval $[a, b]$ so that the basis functions will not be orthonormal. Consequently, we should increase the number of terms used in the regressions.

## 4.1 Valuation of the Standard Put Option

The first derivative we price is an American put option on a non-dividend stock with the following characteristics: $\sigma = 0.2$, $r = 0.06$, $T = 1$, $S_0 = E = 40$. We approximate this option assuming that there are 70 exercise dates.

The value of this option using the binomial method of Cox *et al.* (1979) (with 1,000 steps) is 2.31928. The value of the corresponding European option, using simulation, binomial trees, and the Black and Scholes (1973) formula, are 2.06193, 2.06560 and 2.06640, respectively.

To avoid numerical problems, we standardize the option dividing by the strike price and we use double-precision variables. We also employ SVDFIT, a Numerical Recipes routine that performs linear least-squares fits using the singular value decomposition technique.

Results for the LSM algorithm with different basis functions and number of terms are shown in Table 4. We use 100,000 simulations, half of them with antithetic variables. Notice that this implies that we have to store (100,000 $\times$ 70) matrices.

[ Insert Table 4 about here ]

For a given polynomial, we see that option values do not increase monotonically with the number of terms. For up to five terms, option prices typically increase. With more terms, option values can decrease and increase later. This result illustrates the limitations of the convergence criterion derived from Proposition 1 of Longstaff and Schwartz (2001).[9] For example, if we use $T_n(x)$ as basis functions, this rule would suggest that five terms are enough to value the option. In this case, the option price would be 2.30689. However, option values increase again taking 7, 8 or 10 terms.

The method seems to be robust with respect to the type of polynomial used. We see that, fixing the number of terms, we obtain similar option prices for different basis functions.

Notice that the computed option prices are lower than the values obtained with the binomial tree. This is not surprising since we are considering only 70 exercise dates. A more reasonable benchmark is the binomial price of the Bermuda option, 2.31547. We use a binomial tree with 1050 steps in which the option can be exercised every 15 steps. Interestingly, LSM seems to slightly underprice the option.

## 4.2 Option on the maximum of five assets

We now analyze a Bermuda call option on the maximum of five uncorrelated assets. The volatility of the asset returns is 0.2, the interest rate is 0.05, the dividend yield is 0.1, the maturity of the option is three years, and there are

[9] This criterion indicates that, to price accurately the option, we should increase the number of basis functions until the option price no longer increases.

three exercise times per year. The strike price is 100 and the initial assets prices are 100 for the five assets.

This option has been priced by Broadie and Glasserman (1997b), using the stochastic mesh method. They find that the 90% confidence interval for the price of this option is $[26.101, 26.211]$.

Longstaff and Schwartz (2001) value this option using the LSM approach with 19 basis functions: a constant, five Hermite polynomials in the maximum of the five assets, the second to the fifth maximums and their square values, the four products of consecutive pairs of maximums, and the product of the five assets. Using 50,000 paths, their option value is 26.182, which is within the interval given by Broadie and Glasserman (1997b).

As before, we use different basis functions to price this option. In Table 5, we show option prices obtained when the Hermite polynomial is replaced by others, with up to ten terms. This means that we use between 14 and 24 basis functions. We simulate 50,000 + 50,000 antithetic paths.

[ Insert Table 5 about here ]

Using less than two terms, we obtain values which are outside the interval given by Broadie and Glasserman (1997b). We also obtain values outside this interval using more than seven terms for the polynomials $P_n(x)$, $H_n(x)$, $T_n(x)$ and $U_n(x)$. In some cases, these prices are lower than 24. Option prices typically increase with up to four or five terms. With more terms, these values can decrease and increase again. As mentioned before, this finding makes it difficult to use the convergence criterion of Longstaff and Schwartz (2001). For example, if we use $L_n(x)$ as basis functions, we would need five terms to price accurately the option. However, the option value increases again with more than six terms.

A final remark is that using five Hermite polynomials, the option price is 26.187, which is very close to the price given by Longstaff and Schwartz (2001).

Note that the value of the corresponding European option is 23.098, so that the early exercise premium is higher than 3.

Now, we set the polynomials equal to $H_{e_n}(x)$, and we change the remaining basis functions. The results are shown in Table 6.

[ Insert Table 6 about here ]

The second column presents the prices obtained without including the squares of the second to the fifth maximums. Dropping out those values has little impact on the option price. As before, the option value increases monotonically only with up to five terms. In the third column, we also leave out the products of consecutive maximums. In this case, option prices are outside the interval except when we use five terms. In the following column, we work with the polynomials $H_{e_n}(x)$, the second to the fifth maximums, and their squares. Now, all the option prices are outside the interval. Finally, the fifth column shows the prices obtained with the same basis functions as in Table 5 plus the third powers of the second to the fifth maximums. Compared to the sixth column of Table 5, we find that option prices are similar in both cases.

## 4.3 American-Bermuda-Asian option

Following Longstaff and Schwartz (2001), we now price a call option on the average of the stock price during a given time horizon. This option can be exercised at any time after some initial period. It matures in two years and it cannot be exercised during the first quarter. The average stock price is the continuous arithmetic mean from three months before the valuation date to time $t$, where $0.25 \leq t \leq 2$.

Table 7 presents the results for 50,000 simulations (25,000 plus 25,000 antithetic) and 100 time steps per year. The third column replicates part of Table 3 in Longstaff and Schwartz (2001). To price the option, we use eight basis functions: a constant, the first two Laguerre polynomials in the stock price, the first two Laguerre polynomials in the average stock price, and the cross products of these Laguerre polynomials up to third degree.

[ Insert Table 7 about here ]

We obtain values slightly below those reported by LOngstaff and Schwartz (2001). For example, when the initial average value of the stock, $A$, is 100 and the underlying stock price, $S$, is 120, our option price is 23.60899 versus 23.775.

The fourth column shows the values computed with different basis functions. We replace Laguerre polynomials with Hermite B polynomials ($H_{e_n}(x)$) of degree five and their cross products up to third degree. We find that changing the type and the degree of the polynomials affects option prices since we now undervalue the option relative to the previous column.[10]

In Table 8, we analyze the sensitivity of the option price with respect to the degree of the Hermite B polynomial.[11] We choose an option ($A = 110,\ S = 120$) and we use up to ten terms in both the underlying stock price and its average. Now, the cross products are not considered.

[ Insert Table 8 about here ]

We see that the value of the option increases with the number of terms up to degree six. Following the convergence criterion derived from Proposition 1 of Longstaff and Schwartz (2001), to price the option, we would use only six terms. However, the value of the option increases again with nine terms. Thus, it is not clear how many terms are really needed.

Finally, we analyze the impact of cross products on the option price in Table 9. We value the previous option with Hermite B polynomials of degrees two and three.

[10]When $A = 100$ and $S = 120$, the option price is 23.48875 versus 23.60899.

[11]We do not report the result for degree one because of numerical difficulties when pricing the option.

[ Insert Table 9 about here ]

In the second row of this table, we do not use cross products. Therefore, these values are taken from Table 8. The third row shows the results using the same cross products as in Longstaff and Schwartz (2001). Apparently, adding cross products does not influence option prices. However, as we can see in the last row of the table, using more cross products seems to affect the option value.[12]

# 5 Conclusions

Monte Carlo simulation is widely used for pricing European options. However, its application for valuing American derivatives is not straightforward.

Recently, Longstaff and Schwartz (2001) have developed the Least-Squares Monte Carlo (LSM) technique, that uses simple regressions to price American options. At each exercise date, they estimate the continuation value of the option regressing the expected cash-flows on basis functions of the underlying asset price.

This paper analyzes the robustness of the LSM approach relative to the type and number of basis functions. We apply this algorithm to price an American put option, a Bermuda call option on the maximum of five assets, and an American-Bermuda-Asian option. We consider ten basis functions with up to ten terms.

Numerical results show that this technique is robust for the simplest (American put) case. For more sophisticated options, the robustness does not seem to be guaranteed and the type and number of basis functions to be used is unclear.

[12]To compute these cross products, we use all the possible pairs of the basis functions employed in the second row. Thus, there are 11 and 22 functions for degrees two and three, respectively.

# References


[1] Abramowitz, W. and I. Stegun (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York.

[2] Ait-Sahlia, F. (1996). Optimal Stopping and Weak Convergence Methods for Some Problems in Financial Economics. *Ph.D. Dissertation*, Dept. of Operations Research, Stanford University.

[3] Ait-Sahlia, F. and P. Carr (1997). American Options: A Comparison of Numerical Methods. In *Numerical Methods in Finance*, edited by L.C.G. Rogers and D. Talay, Cambridge University Press.

[4] Ait-Sahlia, F. and T.L. Lai (1996). Approximations for American Options, working paper, Cornell University.

[5] ——— (2000). A Canonical Optimal Stopping Problem for American Options and its Numerical Solution. *Journal of Computational Finance*, 3, 2, 33–52.

[6] Barone-Adesi, G. and R.E. Whaley (1987). Efficient Analytic Approximation of American Option Values. *Journal of Finance*, 42, 2, 301–320.

[7] Barraquand, J. and D. Martineau (1995). Numerical Valuation of High Dimensional Multivariate American Securities. *Journal of Financial and Quantitative Analysis*, 30, 383–405.

[8] Black, F. and M. Scholes (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81, 3, 637–654.

[9] Boyle, P. (1977). Options: A Monte Carlo Approach. *Journal of Financial Economics*, 4, 323–338.

[10] Boyle, P. (1988). A Lattice Framework for Option Pricing with Two State Variables. *Journal of Financial and Quantitative Analysis*. 22, 1–12.

[11] Boyle, P., M. Broadie and P. Glasserman (1997). Monte Carlo Methods for Security Pricing. *Journal of Economic Dynamics and Control*, 21, 1267–1321.

[12] Breen, R. (1991). The Accelerated Binomial Option Pricing Model. *Journal of Financial and Quantitative Analysis*, 26, 153–164.

[13] Brennan, M.J. and E.S. Schwartz (1977). The Valuation of American Put Options. *Journal of Finance*, 32, 449–462.

[14] ——— (1978). Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis. *Journal of Financial and Quantitative Analysis*, 13, 461–474.

[15] Broadie, M. and J. Detemple (1996). American Option Valuation: New Bounds, Approximations and a Comparison of Existing Methods. *Review of Financial Studies*, 9, 1211–1250.

[16] Broadie, M. and P. Glasserman (1997a). Pricing American-Style Securities using Simulations. *Journal of Economic Dynamics and Control*, 21, 1323–1352.

[17] ——— (1997b). A Stochastic Mesh Method for Pricing High-Dimensional American Options, working paper, Columbia University.

[18] Broadie, M., P. Glasserman and G. Jain (1997). Enhanced Monte Carlo estimation for American Option Prices. Journal of Derivatives, 5, 1, 25-44.

[19] Bunch, D. and H.E. Johnson (1992). A Simple and Numerically Efficient Valuation Method for American Puts Using a Modified Geske-Johnson Approach. *Journal of Finance*, 47, 2, 809–816.

[20] ——— (2000). The American Put Option and Its Critical Stock Price. *Journal of Finance*, 55, 5, 2333–2356.

[21] Carr, P. (1998). Randomization and the American Put. *Review of Financial Studies*, 11, 597–626.

[22] Carr, P. and D. Faguet (1996). Fast Accurate Valuation of American Options, working paper, Cornell University.

[23] Carr, P., R. Jarrow and R. Mynemi (1992). Alternative Characterization of American Puts. *Mathematical Finance*, 2, 87–106.

[24] Clewlow, L. and C. Strickland (1998), *Implementing Derivatives Models*, John Wiley & Sons Ltd., England.

[25] Courtadon (1982). A More Accurate Finite Difference Approximation for the Valuation of Options. *Journal of Financial and Quantitative Analysis*, 17, 5, 697–703.

[26] Cox, J.C., S.A. Ross and M. Rubinstein (1979). Option Pricing: A Simplified Approach. *Journal of Financial Economics*, 7, 3, 229–263.

[27] Demidowitsch, B.P., I.A. Maron and E.S. Schuwalowa (1980). Métodos numéricos de análisis. Ed. Paraninfo, S.A. Madrid.

[28] Geske, R. (1979). A Note on an Analytical Valuation Formula for Unprotected American Options on Stocks with Known Dividends. *Journal of Financial Economics*, 7, 375–380.

[29] Geske, R. and H.E. Johnson (1984). The American Put Option Valued Analytically. *Journal of Finance*, 39, 5, 1511–1524.

[30] Harrison, J.M. and D. Kreps (1979). Martingales and Arbitrage in Multiperiod Securities Markets. *Journal of Economic Theory*, 20, 3, 381–408.

[31] Harrison, J.M. and S. Pliska (1981). Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stochastic Processes and their Applications*, 11, 215–260.

[32] Ho, T.S.Y., R.C. Stapleton and M.G. Subrahmanyam (1994). A Simple Technique for the Valuation and Hedging of American Options. *Journal of Derivatives*, 2, 52–66.

[33] ——— (1997). The Valuation of American Options with Stochastic Interest Rates: A Generalization of the Geske-Johnson Technique. *Journal of Finance*, 52, 2, 827–840.

[34] Huang, J.Z., M.G. Subrahmanyam and G.G. Yu (1996). Pricing and Hedging American Options: A Recursive Integration Method. *Review of Financial Studies*, 9, 1, 277–300.

[35] Hutchinson, J., A. Lo and T. Poggio (1994). A Nonparametric Approach to Pricing and Hedging Derivative Securities. *Journal of Finance*, 49, 3, 851–886.

[36] Ibañez, A. and F. Zapatero (1998). Monte Carlo Valuation of American Options through Computation of the Optimal Exercise Frontier, working paper, The University of Southern California.

[37] Jacka, S.D. (1991). Optimal Stopping and the American Put. *Mathematical Finance*, 1, 1–14.

[38] Johnson, H.E. (1983). An Analytic Approximation for the American Put Price. *Journal of Financial and Quantitative Analysis*, 18, 1, 141–148.

[39] Ju, N. (1998). Pricing American Option by Approximating its Early Exercise Boundary as a Multipiece Exponential Function. *Review of Financial Studies*, 11, 3, 627–646.

[40] Ju, N. and R. Zhong (1999). An Approximate Formula for Pricing American Options. *The Journal of Derivatives*, 7, 2, 31–40.

[41] Kim, I.J. (1990). The Analytical Valuation of American Options. *Review of Financial Studies*, 3, 547–572.

[42] Longstaff, F.A. and E.S. Schwartz (2001). Valuing American Options by Simulations: A Simple Least-Squares Approach. *Review of Financial Studies*, 14, 1, 113–147.

[43] McKean, H.P. (1965). Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem in Mathematical Economics. *Industrial Management Review*, 6, 32–39.

[44] Merton, R.C. (1973). Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science*, 4, 1, 141–183.

[45] Morton, K.W. and D.F. Mayers (1998). *Numerical Solution of Partial Differential Equations*. Cambridge University Press.

[46] Parkinson, M. (1977). Option Pricing: The American Put. *Journal of Business, 50*, 21–36.

[47] Raymar, S. and M. Zwecher (1997). Monte Carlo Estimation of American Call Options on the Maximum of Several Stocks. *Journal of Derivatives*, 5, 7–24.

[48] Rektorys, K. (1982). *The Method of Discretization in Time and Partial Differential Equations*, D. Reidel Publishing, Boston, Mass.

[49] Rendleman, R. and B. Bartter (1979). Two-State Option Pricing. *Journal of Finance*, 34, 1093–1110.

[50] Roll, R. (1977). An Analytic Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends. *Journal of Financial Economics, 5*, 251–258.

[51] Schwartz, E.S. (1977). The Valuation of Warrants: Implementing a New Approach. *Journal of Financial Economics*, 4, 79–93.

[52] Tilley, J. (1993). Valuing American Options in a Path Simulation Model. *Transactions of the Society of Actuaries*, 45, 83–104.

[53] Whaley, R.E. (1981). On the Valuation of American Call Options on Stocks whit Known Dividends. *Journal of Financial Economics*, 9, 2, 207–211.

Table 1: **Explicit expressions of the basis functions.**

| $f_n(x)$ | $N$ | $d_n$ | $c_m$ | $g_m(x)$ |
|---|---|---|---|---|
| $W_n(x)$ | $0$ | $1$ | $1$ | $x^n$ |
| $P_n(x)$ | $[n/2]$ | $2^{-n}$ | $(-1)^m \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} 2n-2m \\ n \end{pmatrix}$ | $x^{n-2m}$ |
| $L_n(x)$ | $n$ | $1$ | $\frac{(-1)^m}{m!} \begin{pmatrix} n \\ n-m \end{pmatrix}$ | $x^m$ |
| $H_n(x)$ | $[n/2]$ | $n!$ | $(-1)^m \frac{1}{m!\,(n-2m)!}$ | $(2x)^{n-2m}$ |
| $H_{e_n}(x)$ | $[n/2]$ | $n!$ | $(-1)^m \frac{1}{m!\,(n-2m)!}$ | $x^{n-2m}$ |
| $T_n(x)$ | $[n/2]$ | $n/2$ | $(-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!}$ | $(2x)^{n-2m}$ |
| $C_n(x)$ | $[n/2]$ | $n$ | $(-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!}$ | $x^{n-2m}$ |
| $T_n^*(x)$ | $[n/2]$ | $2^{-n}\ n$ | $(-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!}$ | $(2x)^{n-2m}$ |
| $U_n(x)$ | $[n/2]$ | $1$ | $(-1)^m \frac{(n-m)!}{m!\,(n-2m)!}$ | $(2x)^{n-2m}$ |
| $S_n(x)$ | $[n/2]$ | $1$ | $(-1)^m \frac{(n-m)!}{m!\,(n-2m)!}$ | $x^{n-2m}$ |

The basis functions are particular cases of the following expression

$$f_n(x) = d_n \sum_{m=0}^{N} c_m g_m(x),$$

where $n \geq 0$ denotes the degree of the polynomial.

Table 2: **Expressions of the basis functions using Rodrigues' formula**.

| $f_n(x)$ | $a_n$ | $\rho(x)$ | $g(x)$ |
|---|---|---|---|
| $W_n(x)$ | $\frac{(2n)!}{n!}$ | $x^{2n}$ | $1$ |
| $P_n(x)$ | $(-1)^n\ 2^n\ n!$ | $1$ | $1-x^2$ |
| $L_n(x)$ | $n!$ | $e^{-x}$ | $x$ |
| $H_n(x)$ | $(-1)^n$ | $e^{-x^2}$ | $1$ |
| $H_{e_n}(x)$ | $(-1)^n$ | $e^{-x^2/2}$ | $1$ |
| $T_n(x)$ | $(-1)^n\ 2^n\ \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$ | $(1-x^2)^{-1/2}$ | $1-x^2$ |
| $C_n(x)$ | $(-1)^n\ 2^n\ \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$ | $\left(1-\frac{x^2}{4}\right)^{-1/2}$ | $1-\frac{x^2}{4}$ |
| $T_n^*(x)$ | $(-1)^n\ 2^{2n-1}\ \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$ | $(1-x^2)^{-1/2}$ | $1-x^2$ |
| $U_n(x)$ | $\frac{(-1)^n\ 2^{n+1}\ \Gamma\left(n+\frac{3}{2}\right)}{(n+1)\ \sqrt{\pi}}$ | $(1-x^2)^{1/2}$ | $1-x^2$ |
| $S_n(x)$ | $\frac{(-1)^n\ 2^{n+1}\ \Gamma\left(n+\frac{3}{2}\right)}{(n+1)\ \sqrt{\pi}}$ | $\left(1-\frac{x^2}{4}\right)^{1/2}$ | $1-\frac{x^2}{4}$ |

The basis functions are especial cases of Rodrigues' formula which is given by

$$f_n(x) = \frac{1}{a_n\ g(x)}\ \frac{\partial^n}{\partial x^n}\ [\rho(x)\ (g(x))^n],$$

where $n \geq 0$ denotes the degree of the polynomial.

Table 3: **Recurrence law for the basis functions**.

| $f_n(x)$ | $a_{n+1}$ | $a_n$ | $b_n$ | $a_{n-1}$ | $f_0(x)$ | $f_1(x)$ |
|---|---|---|---|---|---|---|
| $W_n(x)$ | 1 | 0 | 1 | 0 | 1 | $x$ |
| $P_n(x)$ | $n+1$ | 0 | $2n+1$ | $n$ | 1 | $x$ |
| $L_n(x)$ | $n+1$ | $2n+1$ | $-1$ | $n$ | 1 | $1-x$ |
| $H_n(x)$ | 1 | 0 | 2 | $2n$ | 1 | $2x$ |
| $H_{e_n}(x)$ | 1 | 0 | 1 | $n$ | 1 | $x$ |
| $T_n(x)$ | 1 | 0 | 2 | 1 | 1 | $x$ |
| $C_n(x)$ | 1 | 0 | 1 | 1 | 2 | $x$ |
| $T_n^*(x)$ | 1 | 0 | 1 | 1/4 | 1 | $x$ |
| $U_n(x)$ | 1 | 0 | 2 | 1 | 1 | $2x$ |
| $S_n(x)$ | 1 | 0 | 1 | 1 | 1 | $2x$ |

The general expression for the recurrence law is given by

$$a_{n+1}\ f_{n+1}(x) = (a_n + b_n\ x)\ f_n(x) - a_{n-1}\ f_{n-1}(x),$$

where $n \geq 0$ denotes the degree of the polynomial.

Table 4: **American put option prices**.

| Number of terms | $W_n(x)$ | $P_n(x)$ | $L_n(x)$ | $H_n(x)$ | $H_{e_n}(x)$ | $T_n(x)$ | $C_n(x)$ | $T_n^*(x)$ | $U_n(x)$ | $S_n(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.09671 | 2.09671 | 2.11487 | 2.09671 | 2.09671 | 2.09671 | 2.09671 | 2.09671 | 2.09671 | 2.09671 |
| 2 | 2.26853 | 2.26853 | 2.29251 | 2.26864 | 2.26853 | 2.26853 | 2.26828 | 2.26853 | 2.26864 | 2.26853 |
| 3 | 2.29770 | 2.29880 | 2.29801 | 2.29892 | 2.29848 | 2.29903 | 2.29628 | 2.29912 | 2.29822 | 2.29848 |
| 4 | 2.30739 | 2.30620 | 2.30761 | 2.30644 | 2.30680 | 2.30624 | 2.30546 | 2.30652 | 2.30703 | 2.30651 |
| 5 | 2.30777 | 2.30677 | 2.30898 | 2.30780 | 2.30816 | 2.30689 | 2.30780 | 2.30744 | 2.30744 | 2.30809 |
| 6 | 2.30803 | 2.30690 | 2.30926 | 2.30812 | 2.30650 | 2.30634 | 2.30646 | 2.30647 | 2.30760 | 2.30670 |
| 7 | 2.30817 | 2.30803 | 2.30818 | 2.30755 | 2.30694 | 2.30720 | 2.30758 | 2.30734 | 2.30761 | 2.30705 |
| 8 | 2.30544 | 2.30791 | 2.30989 | 2.30814 | 2.30805 | 2.30854 | 2.30765 | 2.30735 | 2.30776 | 2.30622 |
| 9 | 2.30585 | 2.30899 | 2.30802 | 2.30806 | 2.30770 | 2.30795 | 2.30728 | 2.30784 | 2.30838 | 2.30696 |
| 10 | 2.30779 | 2.30760 | 2.30877 | 2.30750 | 2.30708 | 2.30800 | 2.30709 | 2.30785 | 2.30828 | 2.30805 |

The parameters of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 1$ year, $S_0 = E = 40$. The value of the American option using the binomial method with 1,000 steps is 2.31928. The values of the corresponding European option, using 100,000 (50,000 plus 50,000 antithetic) simulations, binomial trees, and the Black and Scholes (1973) formula, are 2.06193, 2.06560, and 2.06640, respectively. We approximate the American option considering 70 exercise dates. For this Bermuda option, we also use a binomial tree, obtaining a value of 2.31547. The first row shows the ten basis functions we use. $n \geq 0$ denotes the degree of the polynomial.

Table 5: **Bermuda call option on the maximum of five assets**.

| Number of terms | $W_n(x)$ | $P_n(x)$ | $L_n(x)$ | $H_n(x)$ | $H_{e_n}(x)$ | $T_n(x)$ | $C_n(x)$ | $T_n^*(x)$ | $U_n(x)$ | $S_n(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | 25.7307 | | | | |
| 1 | 25.87009 | 25.87009 | 25.68810 | 25.87009 | 25.87009 | 25.87009 | 25.87009 | 25.87009 | 25.87009 | 25.87009 |
| 2 | 26.11208 | 26.11183 | 26.03520 | 26.11600 | 26.10928 | 26.11103 | 26.10783 | 26.11108 | 26.11175 | 26.10928 |
| 3 | 26.17662 | 26.17663 | 26.16028 | 26.17832 | 26.17398 | 26.17694 | 26.17562 | 26.17893 | 26.17624 | 26.17475 |
| 4 | 26.19315 | 26.19654 | 26.17243 | 26.19629 | 26.19432 | 26.19424 | 26.18763 | 26.19282 | 26.20281 | 26.19682 |
| 5 | 26.19880 | 26.19532 | 26.17635 | 26.18700 | 26.19673 | 26.18403 | 26.19299 | 26.20364 | 26.18410 | 26.19638 |
| 6 | 26.19278 | 26.19700 | 26.17094 | 26.18636 | 26.19645 | 26.18781 | 26.18967 | 26.19001 | 26.16213 | 26.18976 |
| 7 | 26.18850 | 26.18622 | 26.17779 | 26.15822 | 26.19626 | 26.11228 | 26.18348 | 26.18593 | 26.11029 | 26.18915 |
| 8 | 26.18072 | 26.08430 | 26.17978 | 26.06568 | 26.19068 | 25.99370 | 26.18848 | 26.18300 | 25.89268 | 26.18736 |
| 9 | 26.17552 | 25.87180 | 26.18153 | 25.97293 | 26.17984 | 25.87901 | 26.18267 | 26.18609 | 25.68113 | 26.18558 |
| 10 | 26.14007 | 25.49381 | 26.18100 | 25.83938 | 26.17448 | 23.89298 | 26.18046 | 26.16613 | 23.99330 | 26.17449 |

The characteristics of the option are: $\sigma = 0.2$ (for the five assets), $r = 0.05$, the dividend yield is 0.1, $T = 3$ years, $E = 100$, there are three exercise times per year, and the initial assets prices are 100 for the five assets. We use 100,000 (50,000 plus 50,000 antithetic) simulations and the following basis functions: a constant, the second to the fifth maximums and their squares, the four products of consecutive pairs of maximums, the product of the five assets, and zero to ten terms of the basis functions indicated in the first row. $n \geq 0$ denotes the degree of the polynomial.

Table 6: **Effect of basis functions on the Bermuda call option on the maximum of five assets**.

| Number of terms | Case I | Case II | Case III | Case IV |
|---|---|---|---|---|
| 0 | 25.75969 | 25.66929 | 25.36842 | 25.76960 |
| 1 | 25.86988 | 25.76343 | 25.84218 | 25.88012 |
| 2 | 26.07051 | 26.03418 | 26.05393 | 26.10515 |
| 3 | 26.15420 | 26.06087 | 26.06920 | 26.18469 |
| 4 | 26.16086 | 26.08385 | 26.09032 | 26.19798 |
| 5 | 26.16341 | 26.10152 | 26.08691 | 26.20064 |
| 6 | 26.14974 | 26.08556 | 26.07915 | 26.18932 |
| 7 | 26.15517 | 26.09288 | 26.07881 | 26.19892 |
| 8 | 26.14555 | 26.09031 | 26.07253 | 26.18422 |
| 9 | 26.14921 | 26.08191 | 26.07572 | 26.17811 |
| 10 | 26.11754 | 26.08579 | 26.07383 | 26.18058 |

The characteristics of the option are: $\sigma = 0.2$ (for the five assets), $r = 0.05$, the dividend yield is 0.1, $T = 3$ years, $E = 100$, there are three exercise times per year, and the initial assets prices are 100 for the five assets. We use 100,000 (50,000 plus 50,000 antithetic) simulations. In Case I we use the following basis functions: a constant, the second to the fifth maximums, the four products of consecutive pairs of maximums, and the product of the five assets. Case II is Case I without the products of consecutive maximums. Case III considers the second to the fifth maximums and their squares. Finally, Case IV uses a constant, the second to the fifth maximums, their squares, their third powers, the products of consecutive pairs of maximums, and the product of the five assets. In all the cases, we also use the polynomials $H_{e_n}(x)$ with up to ten terms.

Table 7: **American-Bermuda-Asian option prices with Laguerre and Hermite B polynomials.**

| $A$ | $S$ | Laguerre | Hermite B |
|---|---|---|---|
| 90 | 100 | 7.72344 | 7.64979 |
| 90 | 110 | 14.18221 | 14.13072 |
| 90 | 120 | 22.15082 | 22.04639 |
| 100 | 100 | 8.29836 | 8.23392 |
| 100 | 110 | 15.37693 | 15.31818 |
| 100 | 120 | 23.60899 | 23.48875 |
| 110 | 100 | 9.45722 | 9.41530 |
| 110 | 110 | 17.12380 | 17.03744 |
| 110 | 120 | 25.29639 | 25.20921 |

The characteristics of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 2$, $E = 100$. The initial value of the average and the underlying stock price are denoted by $A$ and $S$, respectively. The average stock price is the continuous arithmetic mean from three months before the valuation date to time $t$, where $0.25 \leq t \leq 2$. The option cannot be exercised during the first quarter. We use 50,000 (25,000 plus 25,000 antithetic) simulations and approximate the American option considering 100 exercise dates per year. We use a constant, the first two Laguerre polynomials in the stock price, the first two Laguerre polynomials in the average stock price, and the cross products of these Laguerre polynomials up to third degree.

Table 8: **Sensitivity of the American-Bermuda-Asian option prices with respect to the degree of Hermite B polynomials.**

| Degree | Option Price |
|---|---|
| 2 | 25.16948 |
| 3 | 25.17115 |
| 4 | 25.19076 |
| 5 | 25.21053 |
| 6 | 25.22129 |
| 7 | 25.15755 |
| 8 | 25.19111 |
| 9 | 25.21444 |
| 10 | 25.21348 |

The characteristics of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 2$, $E = 100$. The initial value of the average is $A = 110$ and the underlying stock price is $S = 120$. The average stock price is the continuous arithmetic mean from three months before the valuation date to time $t$, where $0.25 \leq t \leq 2$. The option can only be exercised after the first quarter. We use 50,000 (25,000 plus 25,000 antithetic) simulations and approximate the American option considering 100 exercise dates per year. We use Hermite B polynomials in the stock price and its average as basis functions.

Table 9: **Sensitivity of the American-Bermuda-Asian option prices with respect to cross products of Hermite B polynomials.**

| | Degree 2 | Degree 3 |
|---|---|---|
| Case I | 25.16948 | 25.17115 |
| Case II | 25.16996 | 25.17099 |
| Case III | 25.14129 | 25.26195 |

The characteristics of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 2$, $E = 100$. The initial value of the average is $A = 110$ and the underlying stock price is $S = 120$. The average stock price is the continuous arithmetic mean from three months before the valuation date to time $t$, where $0.25 \leq t \leq 2$. The option can only be exercised after the first quarter. We use 50,000 (25,000 plus 25,000 antithetic) simulations and approximate the American option considering 100 exercise dates per year. We use Hermite B polynomials and their cross products in the stock price and its average as basis functions. In Case I, we do not use cross products. Case II employs the same cross products as in Longstaff and Schwartz (2001). In Case III, we use all the possible pairs of the basis functions employed in Case I.