

AUSTRALIAN ASIAN OPTIONS

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Abstract

We study European options on the ratio of the stock price to its average and viceversa. Some of these options are traded in the Australian Stock Exchange since 1992, thus we call them Australian Asian options. For geometric averages, we obtain closed-form expressions for option prices. For arithmetic means, we use different approximations that produce very similar results.

Keywords

Asian Options, Arithmetic Average, Geometric Average, Edgeworth Expansion, Lognormal Distribution, Gamma Distribution

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Australian Asian Options

Abstract

We study European options on the ratio of the stock price to its average and viceversa. Some of these options are traded in the Australian Stock Exchange since 1992, thus we call them Australian Asian options. For geometric averages, we obtain closed-form expressions for option prices. For arithmetic means, we use different approximations that produce very similar results.

1 Introduction

Asian options are options on average values of asset prices. Their prices depend on the asset price history over the averaging period. Therefore, they are considered path-dependent claims.

These derivatives are traded in over-the-counter markets¹ and it is usually argued that they provide the following advantages: (a) they are cheaper than standard European ones as the average is less volatile than the asset price itself, (b) this type of options prevents manipulation of the underlying asset price at the maturity date and (c) they are the adequate hedging instrument for traders who act continuously over finite periods.

Options on the ratio of the stock price to its average (or viceversa) are particular cases of Asian options. They have recently appeared as special types of variable purchase options (VPOs). VPOs were first issued in 1992 and have been traded since then on the Australian Stock Exchange. A VPO is an option that gives its holder the right to buy at maturity a stochastic number of shares that depends on the terminal stock price. This option can have more complex features like caps and floors on the number of shares. Handley (2000) provides a detailed description of VPOs as well as pricing formulae. He also describes Asian VPOs, in which the number of shares that can be bought at maturity depends on the average stock price. These options are shown to be equivalent to options on the ratio of the stock price to its average. Alternatively, we could define Asian VPOs in such a way that they are equivalent to options on the ratio of the average of the stock price to the stock price itself.

¹For several examples in the real life, see Bouaziz *et al* (1994) and Vorst (1996).

In this paper we price such options using both geometric and arithmetic (discrete- and continuous-time) means of stock prices that are assumed to follow a lognormal process.² When the average is computed on geometric basis, these ratios are lognormally distributed at maturity, thus we obtain Black-Scholes-type formulae to price options.

However, when the average is computed on arithmetic basis, the risk-neutral distribution of these ratios is, in general, unknown and we can not obtain closed-form expressions for the prices of these options³. This happens because the arithmetic average is the convolution of correlated lognormal random variables and its distribution is unknown.⁴

Some of the approaches that have been followed in the literature to study the pricing and hedging of arithmetic Asian options are:

- Numerical approximations
 - Finite different schemes. In this case, a partial difference pricing equation is obtained and then solved numerically. See, for instance, Kemna and Vorst (1990), Rogers and Shi (1995), Alziari *et al* (1997) or Hansen and Jorgensen (2000).⁵

²This is one of the main assumptions in the Black-Scholes model. See Armata (2001) or Dinenis *et al* (2001) for valuation of Asian options outside the Black-Scholes framework.

³As will be indicated later, the distribution of the continuous-time average is known, allowing us to obtain exact analytical expressions for option prices.

⁴It may be worth noting that if these variables were perfectly correlated, then the average would be lognormal. Alternatively, if these variables are i.i.d., applying the central limit theorem, the distribution of the average would converge to the normal one.

⁵Other examples are Deynne and Wilmott (1995a, 1995b), He and Takahashi (1995-96), Zvan *et al* (1998) or Shreve and Večer (2000).

- Monte Carlo simulations. Here, variance reductions technique are commonly used to reduce standard errors. For example, Kemna and Vorst (1990) use the closed-form expressions for prices of geometric Asian options as control variables to price arithmetic ones with Monte Carlo.⁶
- General numerical methods. Some examples are the fast Fourier transform proposed in Carverhill and Clewlow (1990), the conditioning approach suggested in Curran (1994), Rogers and Shi (1995) and Nielsen and Sandmann (1996, 1999, 2001), and the accelerated simulation method presented in Vázquez-Abad and Dufresne (1998).
- Pseudo-analytic characterizations. this is the case of Yor (1992, 1993), Geman and Yor (1993), De Schepper *et al* (1994), Eydeland and Geman (1995), Fu *et al* (1999) and Shirakawa (1999), who use the theory of Bessel processes and the inversion of a Laplace transform. Alternatively, Ju (1997) employs A Fourier transform, while Dufresne (2000) suggests a Laguerre expansion.
- Analytical approximations. Jarrow and Rudd (1982) apply Edgeworth series expansion to the problem of option pricing when the risk-neutral distribution of the underlying asset at maturity is not known. Typically, up to fourth order moments are used to approximate the true probability density function. This method has been also used by Turnbull and Wakeman (1991), Ritchken *et al* (1993) and Jacques (1996). Some authors use only the first and second order moments in the Edge-

⁶See also Haykov (1993), Corwin *et al* (1996) and Nielsen and Sandmann (1996).

worth series expansion, obtaining what is called the Wilkinson approximation.⁷ This is equivalent to assume that the true distribution is actually lognormal, so that Black-Scholes-type formulae can be used to price options. See Levy and Turnbull (1992) for a numerical comparison of the accuracy of these expansions. Other analytical approximations can be seen in Bouaziz *et al* (1994), Vorst (1990, 1992, 1996), Nielsen and Sandmann (1998), Posner and Milevsky (1998) and Chung *et al* (2001).

A different type of analytical approximation is presented in Milevsky and Posner (1998) who use the fact that the infinite sum of correlated lognormal random variables is reciprocal gamma distributed to obtain a closed-form solution for the value of arithmetic Asian options.⁸ This formula is exact only when the average is computed continuously.

In this paper we price arithmetic Australian Asian options using both the Wilkinson approximation and the gamma distribution. We also use Monte Carlo simulation with antithetic variables. The results show that option prices obtained with the three methods are quite similar. This is true even when the number of monitoring dates used to compute the average is small. Hence, in practice, it does not seem to be necessary to use higher order moments in the Edgeworth expansion nor to require a large number of monitoring dates in the gamma approximation to the option pricing problem.

The rest of the paper is organized as follows. Section 2 describes some

⁷See, for example, Levy (1992), Hansen and Jorgensen (2000) and Dinenis *et al* (2001).

⁸Merton (1975) and Majumdar and Radner (1991) are examples of papers in the economic literature that deal with the gamma distribution.

statistical results that are used in the paper. In Section 3 we generalize the Black-Scholes formula for option prices. Section 4 presents closed-form expressions for the prices of geometric Australian Asian options and Section 5 presents approximations to the value of arithmetic ones. Finally, Section 6 summarizes and concludes.

2 Previous Statistical Results

The following Lemma specifies several features of the lognormal distribution that will be useful to find later results:

Lemma 1

1. Let $Y = \ln(X)$ be a normal random variable with mean m and variance s^2 . Then, X follows a lognormal distribution, that is, $X \sim \Lambda(m, s^2)$.

Its density function is given by

$$f(x) = \frac{1}{sx\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln x - m}{s}\right)^2\right\}, \quad x > 0 \quad (1)$$

Moreover, it is verified that

$$E(X) = \exp\left\{m + \frac{1}{2}s^2\right\} \quad (2)$$

$$V(X) = [E(X)]^2 [e^{s^2} - 1] \quad (3)$$

$$E(X^{-1}) = \exp\{-2m\} E(X) \quad (4)$$

$$V(X^{-1}) = [E(X^{-1})]^2 [e^{s^2} - 1] \quad (5)$$

2. *The expectation of the truncated lognormal variable*

$$\tilde{X} = \begin{cases} X & \text{if } X \geq K \\ 0 & \text{if } X < K \end{cases}, \quad K \in \mathbb{R}^+$$

is given by

$$E(\tilde{X}) = E(X) N(s - D), \quad D = \frac{\ln K - m}{s}$$

where $N(\cdot)$ denotes the distribution function of a standard normal variable.

Proof: See Appendix.

The following Lemma will be useful to compute the moments of ratios involving arithmetic average asset prices:

Lemma 2 (*Mood et al (1974)*)

Let X and Y be two random variables. Then, it is verified that

$$\begin{aligned} E\left(\frac{X}{Y}\right) &\simeq \frac{E(X)}{E(Y)} - \frac{1}{(E(Y))^2} \text{Cov}(X, Y) + \frac{E(X)}{(E(Y))^3} V(Y) \\ V\left(\frac{X}{Y}\right) &\simeq \left(\frac{E(X)}{E(Y)}\right)^2 \left(\frac{V(X)}{(E(X))^2} + \frac{V(Y)}{(E(Y))^2} - 2\frac{\text{Cov}(X, Y)}{E(X)E(Y)}\right) \end{aligned}$$

Proof: See Mood *et al*, p. 181.

3 The Generalized Black-Scholes Model

Uncertainty is modelled by a filtered probability space (Ω, \mathcal{F}, P) . The set of trading dates is $t \in [0, T]$. Let $Z = \{Z_t \mid t \in [0, T]\}$ be the price process for the underlying asset. The assumptions of the pricing model are as follows:

1. Markets are frictionless
 - (a) No transaction costs when trading the stock or the option
 - (b) No taxes

- (c) No penalties to short selling
 - (d) All assets are perfectly divisible
2. Security trading is in continuous time
 3. The term structure of interest rates is flat and known with certainty.
Let $\beta = \{ \beta_t; t \in [0, T] \}$ be the value process for a banking account defined by

$$d\beta_t = r\beta_t dt, \quad (6)$$

where $t \in [0, T]$, $\beta_0 = 1$ and the risk-free interest rate, r , is constant

4. The asset offers a continuous dividend yield of δ in the interval $[0, T]$
5. The asset price follows a GBM process

$$dZ_t = \mu_Z Z_t dt + \sigma_Z Z_t dW_t, \quad (7)$$

where $(Z_t, t) \in (0, \infty) \times [0, T]$, μ_Z and σ_Z are constants and W_t is a standard Wiener process. Usually, σ_Z^2 is referred to as the logarithmic variance parameter of the asset.

Under the risk-neutral probability measure, the process (7) becomes

$$dZ_t = \alpha_Z Z_t dt + \sigma_Z Z_t dW_t$$

where α_Z is the (constant) risk-neutral drift of the process.

The solution for this process is given by

$$Z_t = Z_0 \exp \left\{ \left(\alpha_Z - \frac{1}{2} \sigma_Z^2 \right) t + \sigma_Z W_t \right\}$$

Therefore, Z_t follows a lognormal process. Moreover, it is straightforward to show that

$$[\ln Z_u] | Z_t \sim N \left(\ln(Z_t) + \left(\alpha_Z - \frac{1}{2} \sigma_Z^2 \right) (u - t), \sigma_Z^2 (u - t) \right), \quad u > t \quad (8)$$

Lemma 3 *The moments of the variable Z_t under the risk-neutral measure are the following:*

$$\begin{aligned} E(Z_t) &= Z_0 e^{\alpha_Z t} \\ V(Z_t) &= [E(Z_t)]^2 [e^{\sigma_Z^2 t} - 1] \\ \text{Cov}(Z_t, Z_s) &= Z_0^2 e^{\alpha_Z (t+s)} [e^{\sigma_Z^2 s} - 1], \quad s < t \end{aligned}$$

Proof: See Appendix.

Obviously, for $s = t$, we obtain

$$\text{Cov}(Z_t, Z_t) = Z_0^2 e^{\alpha_Z (t+t)} [e^{\sigma_Z^2 t} - 1] = Z_0^2 e^{2\alpha_Z t} [e^{\sigma_Z^2 t} - 1] = V(Z_t)$$

The following proposition generalizes the Black-Scholes option pricing formula:

Proposition 1 *The price at time 0 of an European call option on Z that matures at time T and with strike price K is given by*

$$C(Z, 0, T, K) = e^{-rT} E(Z_T) N(d_1) - K e^{-rT} N(d_2) \quad (9)$$

where

$$\begin{aligned} d_1 &= \frac{\ln \left(e^{-\alpha_Z T} E(Z_T) / K \right) + \left(\alpha_Z + \frac{1}{2} \sigma_Z^2 \right) T}{\sigma_Z \sqrt{T}} \\ d_2 &= d_1 - \sigma_Z \sqrt{T} \end{aligned}$$

Proof: See Appendix.

The prices of European put options can be easily obtained using the put-call parity:

$$P(Z, 0, T, K) = C(Z, 0, T, K) - e^{-rT} E(Z_T) + Ke^{-rT} \quad (10)$$

Let S_t denote the stock price at time t . We suppose that S_t follows the risk-neutral process

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t, \quad (11)$$

where q is the continuous dividend yield of the stock and σ is a constant.

The solution for this process is given by

$$S_t = S_0 \exp \left\{ \left(r - q - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\} \quad (12)$$

Note that application of Proposition 1 to the stock price process (11) leads to the Black-Scholes formula adjusted by dividends, as derived by Merton (1973).

4 Geometric Australian Asian Options

We consider n monitoring dates so that the time interval $[0, T]$ is partitioned in the following way:

$$\{t_0 = 0 < t_1 < t_2 < \dots < t_n = T\}, \quad t_i - t_{i-1} = \frac{T}{n} = \Delta t, \quad \forall i = 1, \dots, n.$$

Let $S = \{ S_{t_i} \equiv S_i, \quad i = 0, 1, \dots, n \}$ be the price process for the stock.

We define the geometric mean of the n stock prices S_1, \dots, S_n as

$$G_n = (S_1 \cdots S_n)^{\frac{1}{n}} = \left(\prod_{i=1}^n S_i \right)^{\frac{1}{n}}, \quad G_0 \equiv S_0$$

Using (12), we have

$$G_n = S_0 \exp \left\{ \left(r - q - \frac{1}{2}\sigma^2 \right) \frac{n+1}{2} \Delta t + \frac{\sigma}{n} \sum_{i=1}^n W_{t_i} \right\} \quad (13)$$

Looking at (12) and (13), and using $t_n = n\Delta t$, we have

$$\frac{S_n}{G_n} = \exp \left\{ \left(r - q - \frac{1}{2}\sigma^2 \right) \frac{n-1}{2} \Delta t + \frac{\sigma}{n} \left[n W_{t_n} - \sum_{i=1}^n W_{t_i} \right] \right\} \quad (14)$$

$$\frac{G_n}{S_n} = \exp \left\{ - \left(r - q - \frac{1}{2}\sigma^2 \right) \frac{n-1}{2} \Delta t - \frac{\sigma}{n} \left[n W_{t_n} - \sum_{i=1}^n W_{t_i} \right] \right\} \quad (15)$$

It is clear from (13)-(15) that the geometric average and both ratios are lognormally distributed. Consequently, we can apply Proposition 2 to price options on these assets after computing their moments. We now state a Lemma that will be useful to obtain these prices.

Lemma 4 *Given the Brownian motions W_{t_i} , $i = 1, \dots, n$, it is verified that*

$$V \left(\sum_{i=1}^n W_{t_i} \right) = \frac{(n+1) \left(n + \frac{1}{2} \right) n}{3} \Delta t$$

$$V \left(n W_{t_n} - \sum_{i=1}^n W_{t_i} \right) = \frac{(n-1) \left(n - \frac{1}{2} \right) n}{3} \Delta t$$

Proof: See Appendix.

Proposition 2 *We consider European call options on S_n/G_n and G_n/S_n that mature at time T and with strike price K . The prices at time 0 of these options are given by expression (9), where the expected value and the logarithmic variance of the asset at maturity are given by the following table.⁹*

⁹For completeness, this table includes the geometric average.

Z_n	$E(Z_n)$	$\sigma_Z^2 T$
G_n	$S_0 \exp \left\{ \left(r - q - \frac{n-1}{6n} \sigma^2 \right) \frac{n+1}{2n} T \right\}$	$\frac{(n+1)(n+\frac{1}{2})}{3n^2} \sigma^2 T$
S_n/G_n	$\exp \left\{ \left(r - q - \frac{n+1}{6n} \sigma^2 \right) \frac{n-1}{2n} T \right\}$	$\frac{(n-1)(n-\frac{1}{2})}{3n^2} \sigma^2 T$
G_n/S_n	$\exp \left\{ - \left(r - q - \frac{5n-1}{6n} \sigma^2 \right) \frac{n-1}{2n} T \right\}$	$\frac{(n-1)(n-\frac{1}{2})}{3n^2} \sigma^2 T$

Proof: See Appendix.

The expression obtained in this Proposition is the Black-Scholes formula with a volatility parameter σ_Z and a continuous dividend yield $\delta = r - \alpha_{Z_n}$ as given by the next table:

Z_n	δ
G_n	$\left(\frac{n-1}{n+1} r + q + \frac{n-1}{6n} \sigma^2 \right) \frac{n+1}{2n}$
S_n/G_n	$\left(\frac{n+1}{n-1} r + q + \frac{n+1}{6n} \sigma^2 \right) \frac{n-1}{2n}$
G_n/S_n	$\left(\frac{3n-1}{n-1} r - q - \frac{5n-1}{6n} \sigma^2 \right) \frac{n-1}{2n}$

Note that the prices of Australian Asian options do not depend on the current stock price, S_0 .

Obviously, when $Z_n = S_n$, we obtain the Black-Scholes formula which does not depend on the partition of the time interval. When $Z_n = G_n$, we get the price derived by Turnbull and Wakeman (1991) and Ritchken *et al* (1993).

The effect of the number of monitoring dates (n) on the expected value of the asset is not clear, since it depends on the relationship among r, q and σ . However, the effect of n on the logarithmic variance, $\sigma_Z^2 T$, is quite obvious as can be seen in Figure 1. The parameter values are $\sigma = 0.2$ and $T = 1$.

[Insert Figure 1 about here]

We observe the following facts:

- The logarithmic variance of the stock price is constant.
- The logarithmic variance of G_n is equal to that of the stock price when $n = 1$. Then it decreases with n , and converges to $\frac{\sigma^2}{3}T$, the logarithmic variance of the continuous geometric average, obtained by Kemna and Vorst (1990).
- The logarithmic variances of S_n/G_n and G_n/S_n are equal. This is due to the relationship between the variances of a lognormal variable and its reciprocal (see expressions (3)-(5)). This variance increases with n and converges to $\frac{\sigma^2}{3}T$.

It can also be of interest to study the effect of σ and T on option prices. We leave this analysis for the case of continuous-time means.

Tables 1 and 2 show call and put option prices respectively (multiplied by 100) for different cases and monitoring dates. Call prices are computed using Proposition 2. Put prices are obtained applying expression (10). The interest rate is 10 % and the stock dividend yield is 3 %. We include the stock price (S_n) and its geometric average (G_n) as underlying assets as a reference. In both cases, we assume that the initial stock price (S_0) is 1.

[Insert Tables 1 and 2 about here]

We see that the value of the Australian Asian options are relatively similar to those of geometric Asian options.

For one monitoring date **and** $G_n = S_T$, we see that options on G_n have the same value as those on the stock. Moreover, Australian options are equal to options on the unity, and their values are given by $\exp\{-rT\} \max\{1 - K, 0\}$.

Interestingly, option prices do not necessarily increase with the volatility of the stock price (σ) either. This is also true for standard geometric Asian options. For example, from Table 1 we have that when $T = 0.5$, $K = 0.8$, and $n = 1,000$ the call option on G_n has a value of 20.548 and 20.538 for $\sigma = 0.2$ and 0.4, respectively.¹⁰

We see that option prices do not necessarily increase with time to maturity (T). For example, when $\sigma = 0.2$, $K = 0.8$, and $n = 100$, Table 1 shows that the call option on G_n/S_n has a value of 18.161 and 16.580 for maturities of 0.5 and 1.0 years, respectively.

The tables show that option prices do not change monotonically with n . For instance, in Table 1 we have that when $\sigma = 0.2$, $T = 0.5$, and $K = 1.1$, call options prices on G_n are 3.175, 0.905, and 0.737, when $n = 1, 10$ and 100, respectively.

The effect of a change in the exercise price (K) is as expected: call prices decrease and put prices increase with K .

As additional reference, the Black-Scholes call option prices (dividend yield = 0) in the four cases studied in Table 1 are 24.027, 27.993, 26.081, and 3.743, and the Black-Scholes put option prices corresponding to Table 2 are 3.400, 3.753, 8.703, and 8.378, respectively.

We now define the continuous geometric average of the stock price over the interval $[0, T]$ as

$$G_T = \exp \left\{ \frac{1}{T} \int_0^T \ln(S_t) dt \right\}$$

¹⁰Recall that these prices have been multiplied by 100.

Using (12), we have

$$G_T = S_0 \exp \left\{ \frac{1}{2} \left(r - q - \frac{1}{2} \sigma^2 \right) T + \frac{\sigma}{T} \int_0^T W_t dt \right\} \quad (16)$$

Looking at (12) and (16), we have

$$\frac{S_T}{G_T} = \exp \left\{ \frac{1}{2} \left(r - q - \frac{1}{2} \sigma^2 \right) T + \frac{\sigma}{T} \left[T W_T - \int_0^T W_t dt \right] \right\} \quad (17)$$

$$\frac{G_T}{S_T} = \exp \left\{ -\frac{1}{2} \left(r - q - \frac{1}{2} \sigma^2 \right) T - \frac{\sigma}{T} \left[T W_T - \int_0^T W_t dt \right] \right\} \quad (18)$$

We now state a Lemma that will be useful to compute the prices of options on these assets.

Lemma 5 *Given the Brownian motions W_t , $t \in [0, T]$, it is verified that*

$$V \left(\int_0^T W_t dt \right) = V \left(T W_T - \int_0^T W_t dt \right) = \frac{T^3}{3}$$

Proof: See Appendix.

Proposition 3 *We consider European call options on S_T/G_T and G_T/S_T that mature at time T and with strike price K . The prices at time 0 of these options are given by expression (9), where the expected value and the logarithmic variance of the asset at maturity are given by the following table.¹¹*

Z_T	$E(Z_T)$	$\sigma_Z^2 T$
S_T	$S_0 \exp\{(r - q)T\}$	$\sigma^2 T$
G_T	$S_0 \exp \left\{ \frac{1}{2} \left(r - q - \frac{1}{6} \sigma^2 \right) T \right\}$	$\frac{\sigma^2}{3} T$
S_T/G_T	$\exp \left\{ \frac{1}{2} \left(r - q - \frac{1}{6} \sigma^2 \right) T \right\}$	$\frac{\sigma^2}{3} T$
G_T/S_T	$\exp \left\{ -\frac{1}{2} \left(r - q - \frac{5}{6} \sigma^2 \right) T \right\}$	$\frac{\sigma^2}{3} T$

¹¹For completeness, this table includes the stock price and its geometric average. The formula for the geometric Asian option was first derived by Kemna and Vorst (1990).

Proof: See Appendix.

The option pricing formula given in this Proposition corresponds to a continuous dividend yield $\delta = r - \alpha_{Z_T}$ as given by the next table:

Z_T	δ
S_T	q
$G_T, S_T/G_T$	$\frac{1}{2} \left(r + q + \frac{1}{6}\sigma^2 \right)$
G_T/S_T	$\frac{1}{2} \left(3r - q - \frac{5}{6}\sigma^2 \right)$

Notice that:

- The logarithmic variances of both ratios are equal to the one derived by Kemna and Vorst (1990) for the continuous geometric average. The intuition for this result is that, with infinite monitoring dates, the volatility of the ratio depends only on the volatility of the average. This value increases with σ and is one third of the variance in the Black-Scholes formula.
- The expected value of G_T is S_0 times the expected value of S_T/G_T .
- The expected values of S_T/G_T and G_T/S_T do not depend on the current stock price, S_0 .
- The expected values of G_T and S_T/G_T are smaller than that of S_T .

Figure 2 shows $E(Z_T)$ as a function of σ . The parameter values are $r = 10\%$, $q = 3\%$, $T = 1$. We assume $S_0 = 1.2$.

[Insert Figure 2 about here]

We observe that the expected values of G_T and S_T/G_T decrease with σ , while the expected value of G_T/S_T increases with σ .¹²

Since the logarithmic variance of the assets studied increase with σ , and their expected values also depend on σ , we have that option prices can decrease with volatility or time to maturity. This surprising result is analyzed next with more detail.

a) Theta

To study the effect of a change in T on option prices, we compute the corresponding partial derivative. Replacing the expected value and the logarithmic variance of the asset at maturity in (9) as given in Proposition 1 and differentiating with respect to T , we get

$$\frac{\partial C(\cdot)}{\partial T} = e^{-rT} \left[(\alpha_{Z_T} - r)E(Z_T)N(d_1) + KrN(d_2) + \frac{\sigma_Z}{2\sqrt{T}}E(Z_T)N'(d_1) \right] \quad (19)$$

If $q > 0$, we have that $\alpha_{Z_T} - r < 0$ for all the assets except for G_T/S_T . Consequently, the effect of T on call options on these assets is undetermined. If $q = 0$, we have that $\alpha_{s_T} - r = 0$, and the stock call price increases with T . For the ratio G_T/S_T , we have that $\alpha_{Z_T} - r > 0 \Leftrightarrow r < \frac{1}{3} \left(q + \frac{5}{6}\sigma^2 \right)$. In this case, an increase in T leads to a higher call option price.

To analyze the effect of an increase of T on the put price, we use the

¹²It can be shown that for σ large enough ($\sigma > \sqrt{6/5}\sqrt{2T\ln(S_0) + 3(r - q)}$), this expected value is higher than $E(S_T)$.

put-call parity (10) and the derivative obtained in (19) to get

$$\frac{\partial P(\cdot)}{\partial T} = e^{-rT} \left[(r - \alpha_{Z_T}) E(Z_T) N(-d_1) - KrN(-d_2) + \frac{\sigma_Z}{2\sqrt{T}} E(Z_T) N'(d_1) \right] \quad (20)$$

The sign of this derivative can be positive or negative, depending on the parameter values. For all the assets except G_T/S_T , we have $r - \alpha_{Z_T} \geq 0$. Then, for these assets, the effect of T on the put price depends on how large is the second term into brackets. If the exercise price is low enough, the put price will increase with T , while for large strikes the opposite will take place. For the ratio G_T/S_T , we have that $r - \alpha_{Z_T} > 0 \Leftrightarrow r > \frac{1}{3} \left(q + \frac{5}{6} \sigma^2 \right)$. In this case, an increase in T can lead to a higher put price if the exercise price is small.

Figure 3 plots geometric Australian option prices as a function of time to maturity. The averages are computed with infinite monitoring dates. The parameters are: $r = 0.1$, $q = 0.03$, $\sigma = 0.2$, $K = 0.8$ for calls and $K = 1.2$ for puts.

[Insert Figure 3 about here]

We see that, in this case, the price of a call option on S_T/G_T increases with T . However, the price of a call option on G_T/S_T decreases with T . The latter result is due to the fact that $r > \frac{1}{3} \left(q + \frac{5}{6} \sigma^2 \right)$, so that $\alpha_{Z_T} - r < 0$ and $\partial C(\cdot)/\partial T$ can be negative.

Since the exercise price for the put options is relatively high ($K = 1.2$), we see that the price of the put option on S_T/G_T decreases with T . The same occurs for a put option on G_T/S_T when time to maturity is small (between 0 and 0.65 years). For higher T , the put price increases. When $T > 1.5$,

the put price decreases again. Interestingly, if we reduce the exercise price to $K = 1.1$, the put price increases for all T .

b) Vega

Replacing the expected value and the logarithmic variance of the asset at maturity in (9) as given in Proposition 1 and differentiating the resulting expression with respect to σ , we obtain that the effect of a change in σ on the price of a call option is given by

$$\nu_C = \frac{\partial C(\cdot)}{\partial \sigma} = e^{-rT} E(Z_T) \sqrt{T} \left[\frac{\partial \alpha_Z}{\partial \sigma} \sqrt{T} N(d_1) + \frac{\partial \sigma_Z}{\partial \sigma} N'(d_1) \right] \quad (21)$$

The partial derivatives into brackets are:

Z_T	$\frac{\partial \alpha_Z}{\partial \sigma}$	$\frac{\partial \sigma_Z}{\partial \sigma}$
S_T	0	1
G_T	$-\frac{1}{6}\sigma$	$\frac{1}{\sqrt{3}}$
S_T/G_T	$-\frac{1}{6}\sigma$	$\frac{1}{\sqrt{3}}$
G_T/S_T	$\frac{5}{6}\sigma$	$\frac{1}{\sqrt{3}}$

Therefore, we obtain

$$\begin{aligned} \nu_C(S_T) &= \frac{\partial C(\cdot)}{\partial \sigma} = S_0 \sqrt{T} N'(d_1) > 0 \\ \nu_C(G_T) &= \frac{\partial C(\cdot)}{\partial \sigma} = e^{-rT} E(G_T) \sqrt{T} \left[-\frac{1}{6}\sigma \sqrt{T} N(d_1) + \frac{1}{\sqrt{3}} N'(d_1) \right] \\ \nu_C(S_T/G_T) &= \frac{\partial C(\cdot)}{\partial \sigma} = e^{-rT} E(S_T/G_T) \sqrt{T} \left[-\frac{1}{6}\sigma \sqrt{T} N(d_1) + \frac{1}{\sqrt{3}} N'(d_1) \right] \\ \nu_C(G_T/S_T) &= \frac{\partial C(\cdot)}{\partial \sigma} = e^{-rT} E(G_T/S_T) \sqrt{T} \left[\frac{5}{6}\sigma \sqrt{T} N(d_1) + \frac{1}{\sqrt{3}} N'(d_1) \right] > 0 \end{aligned}$$

It is straightforward to show that

$$\begin{aligned}\nu_C(G_T) < 0 &\Leftrightarrow \frac{N'(d_1)}{\sigma N(d_1)} < \frac{1}{2}\sqrt{\frac{T}{3}} \\ \nu_C(S_T/G_T) < 0 &\Leftrightarrow \frac{N'(d_1)}{\sigma N(d_1)} < \frac{1}{2}\sqrt{\frac{T}{3}}\end{aligned}$$

Hence, the vega of a call option on in the Kemna and Vorst (1990) model and the vega of a geometric Australian asian call option on S_T/G_T can be negative. As shown later, this can occur for reasonable parameter values.

To analyze the effect of an increase of σ on the put price, we use the put-call parity (10) and the derivative obtained in (21) to obtain

$$\nu_P = \frac{\partial P(\cdot)}{\partial \sigma} = e^{-rT} E(Z_T) \sqrt{T} \left[\frac{\partial \sigma_Z}{\partial \sigma} N'(d_1) - \frac{\partial \alpha_Z}{\partial \sigma} \sqrt{T} N(-d_1) \right] \quad (22)$$

Therefore, the vega of put options on our assets are:

$$\begin{aligned}\nu_P(S_T) &= \frac{\partial P(\cdot)}{\partial \sigma} = S_0 \sqrt{T} N'(d_1) > 0 \\ \nu_P(G_T) &= \frac{\partial P(\cdot)}{\partial \sigma} = e^{-rT} E(G_T) \sqrt{T} \left[\frac{1}{6} \sigma \sqrt{T} N(-d_1) + \frac{1}{\sqrt{3}} N'(d_1) \right] > 0 \\ \nu_P(S_T/G_T) &= \frac{\partial P(\cdot)}{\partial \sigma} = e^{-rT} E(S_T/G_T) \sqrt{T} \left[\frac{1}{6} \sigma \sqrt{T} N(-d_1) + \frac{1}{\sqrt{3}} N'(d_1) \right] > 0 \\ \nu_P(G_T/S_T) &= \frac{\partial P(\cdot)}{\partial \sigma} = e^{-rT} E(G_T/S_T) \sqrt{T} \left[-\frac{5}{6} \sigma \sqrt{T} N(-d_1) + \frac{1}{\sqrt{3}} N'(d_1) \right]\end{aligned}$$

It is clear that

$$\nu_P(G_T/S_T) < 0 \Leftrightarrow \frac{N'(d_1)}{\sigma N(-d_1)} < \frac{5}{2}\sqrt{\frac{T}{3}}$$

As before, this inequality can hold for reasonable parameter values.

Figure 4 exhibits geometric Australian option prices as a function of volatility (σ). The averages are computed with infinite monitoring dates. The parameters are: $r = 0.1$, $q = 0.03$, $T = 0.1$, $K = 0.8$ for calls and $K = 1.1$ for puts.

[Insert Figure 4 about here]

We see that the price of the call option on S_T/G_T first decreases and then increases with volatility. The vega of this option is zero when $\sigma = 0.67$. As expected, the price of the call option on G_T/S_T and a put option on S_T/G_T always increase with σ . However, the price of the put option on G_T/S_T first decreases and then increases with volatility. The vega of this option is zero when $\sigma = 0.36$.

To summarize the results, both call and put geometric Australian option prices can increase and decrease with either time to maturity or volatility. This is clearly seen in Figure 5, that shows option prices as a function of both variables. The parameter values are: $r = 0.1$, $q = 0.03$, $n = \infty$, $K = 0.8$ for the call option and $K = 1.1$ for the put option.

[Insert Figure 5 about here]

Finally, Tables 1 and 2 present geometric option prices when $n = \infty$. Note that the option on S_n/G_n is equivalent to the option on G_n . This happens because we are taking $S_0 = 1$. We see that option prices for continuous averages are almost identical to those for discrete averages with 1,000 monitoring dates. For example, from Table 2 we have that when $\sigma = 0.2$, $T = 1$, $K = 1.0$ and $n = \infty$, the put options on G_n , S_n/G_n , and G_n/S_n have values of 2.935, 2.935, and 5.002, respectively, while that when $n = 1,000$ those prices are 2.937, 2.933, and 5.000, respectively.

5 Arithmetic Australian Asian Options

We define the discrete arithmetic mean of the n stock prices S_1, \dots, S_n as

$$A_n = \frac{1}{n}(S_1 + \dots + S_n) = \frac{1}{n} \sum_{i=1}^n S_i, \quad A_0 \equiv S_0 \quad (23)$$

The continuous counterpart is given by

$$A_T = \frac{1}{T} \int_0^T S_t dt \quad (24)$$

As mentioned previously, the distribution of A_n is unknown. Therefore, we can not apply Proposition 1 to price options. As described in the following sections, two ways to overcome this problem are:

- To approximate the true distribution with an alternative one.
- To approximate the distribution of A_n with that of A_T .

5.1 Pricing the Options with the Edgeworth / Wilkin-son Approximation

To price options, we approximate the risk-neutral distribution of the underlying asset at maturity with a tractable distribution. We perform this approximation by expanding the true distribution around the approximating one. This approach is called generalized Edgeworth series expansion. The coefficients of this expansion are function of the moments of the true and approximating distribution. Considering up to four terms in this expansion and specifying the approximating distribution to be lognormal, we will show that the (approximate) option price is equal to the Black-Scholes price plus

three adjustment terms. These terms depend, respectively, on the difference between the variance, skewness, and kurtosis of the true and the lognormal distribution. The intuition is that the first four moments of the distribution are enough to reflect the effects of the distribution on option prices.

More concretely, we approximate the true probability distribution, $F(s)$, with an approximating distribution, $A(s)$. It is assumed that both distributions have continuous density functions, $f(s)$ and $a(s)$. We employ the following notation:

$$\begin{aligned}\alpha_j(F) &= \int_{-\infty}^{\infty} s^j f(s) ds \\ \mu_j(F) &= \int_{-\infty}^{\infty} (s - \alpha_1(F))^j f(s) ds \\ \Psi(F, t) &= \int_{-\infty}^{\infty} e^{its} f(s) ds, \quad i = \sqrt{-1}\end{aligned}$$

where $\alpha_j(F)$ and $\mu_j(F)$ are, respectively, the j -th non-central and central moments of F and $\Psi(F, t)$ is the characteristic function of F .¹³

Following Stuart and Ord (1987), the cumulants $k_j(F)$ of the distribution F are defined by the identity in t

$$\exp\left(\sum_{j=1}^{\infty} k_j(F) \frac{t^j}{j!}\right) = \sum_{j=0}^{\infty} \alpha_j(F) \frac{t^j}{j!}$$

or, equivalently

$$\ln \Psi(F, t) = \sum_{j=1}^{\infty} k_j(F) \frac{(it)^j}{j!}$$

For practical purposes, we only need the first four cumulants in the Edgeworth series expansion. These cumulants are, respectively, the mean, the

¹³Analogous notation is employed for the approximating distribution A .

variance, the coefficient of skewness and the excess of kurtosis:

$$\begin{aligned} k_1(F) &= \alpha_1(F), & k_2(F) &= \mu_2(F) \\ k_3(F) &= \mu_3(F), & k_4(F) &= \mu_4(F) - 3\mu_2^2(F) \end{aligned}$$

Jarrow and Rudd (1982) prove the following series expansion for $f(s)$ around $a(s)$:

$$\begin{aligned} f(s) &= a(s) + \frac{k_2(F) - k_2(A)}{2!} \frac{d^2 a(s)}{ds^2} - \frac{k_3(F) - k_3(A)}{3!} \frac{d^3 a(s)}{ds^3} \\ &\quad + \frac{k_4(F) - k_4(A) + 3(k_2(F) - k_2(A))^2}{4!} \frac{d^4 a(s)}{ds^4} + \varepsilon(s) \end{aligned} \quad (25)$$

where, by construction, $k_1(F)$ is set equal to $k_1(A)$.

The difference between $f(s)$ and $a(s)$ depends on the cumulants of both distributions with weighting factors given by the derivatives of $a(s)$. The terms on the right-hand side of (25) reflect any difference in variance, skewness and kurtosis and variance between $f(s)$ and $a(s)$. The residual error, $\varepsilon(s)$, includes any remaining difference.

Now, we employ (25) to obtain an approximate option price. Using $f(s)$ as the true distribution of the asset price at maturity, we obtain the expected value at maturity of an option on this asset. Then, this expansion provides an approximated expected value for the option at maturity in terms of the approximating distribution, $a(s)$.

In a risk-neutral world, the true price of the call option, $C(F)$, is obtained by discounting its expected value at the risk-free rate:

$$C(F) = e^{-rT} \int_{-\infty}^{\infty} \max\{S_T - K, 0\} dF(S_T)$$

Using (25) and a little algebra, this price becomes

$$C(F) = C(A) + e^{-rT} \frac{k_2(F) - k_2(A)}{2!} a(K) - e^{-rT} \frac{k_3(F) - k_3(A)}{3!} \left. \frac{da}{dS_T} \right|_K$$

$$+e^{-rT} \frac{k_4(F) - k_4(A) + 3(k_2(F) - k_2(A))^2}{4!} \frac{d^2 a}{dS_T^2} \Big|_K + \varepsilon(K) \quad (26)$$

where

$$C(A) = e^{-rT} \int_{-\infty}^{\infty} \max\{S_T - K, 0\} dA(S_T)$$

A natural candidate for the approximating distribution is the lognormal one. In this case, $C(A)$ is equal to the Black-Scholes option price.

Equation (26) shows that the true option price is equal to $C(A)$ plus three adjustment terms. As before, these terms correct for difference in variance, skewness and kurtosis and variance between the distributions F and A while the term $\varepsilon(K)$ includes the residual error.

As mentioned in the Introduction, the Wilkinson approximation is a particular case of the Edgeworth expansion, where just the first two cumulants are used.

5.2 Pricing the Options with the Gamma Distribution

It is known that the infinite sum of lognormal distributions is a reciprocal gamma distribution. Using this distribution as state-price density function, Milevsky and Posner (1998) obtain a closed-form expression for the price of arithmetic Asians options. The solution is the same as the Black-Scholes formula where the normal distribution is replaced by the gamma one.

We briefly summarize the main characteristics of the gamma distribution. Let X be gamma distributed with parameters α and β , that is, $X \sim \Gamma(\alpha, \beta)$. Its density function is given by

$$g(x) = \frac{\beta^{-\alpha} x^{\alpha-1} \exp\left\{-\frac{x}{\beta}\right\}}{\Gamma(\alpha)}, \quad x > 0$$

where $\Gamma(x)$ is the gamma function, defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

The mean and variance of the gamma distribution are

$$E(X) = \alpha\beta, \quad V(X) = \alpha\beta^2.$$

If we define $Y = \frac{1}{X}$, then Y follows a reciprocal gamma distribution. Its first two non-central moments are

$$\begin{aligned} M_1 &= E(Y) = \frac{1}{\beta(\alpha - 1)} \\ M_2 &= E(Y^2) = \frac{1}{\beta^2(\alpha - 1)(\alpha - 2)} \end{aligned}$$

The variance is given by

$$V(Y) = M_2 - M_1^2 = \frac{1}{\beta^2(\alpha - 1)^2(\alpha - 2)}$$

It is straightforward to obtain the following relationships:

$$\alpha = \frac{2M_2 - M_1^2}{M_2 - M_1^2}, \quad \beta = \frac{M_2 - M_1^2}{M_1 M_2} \quad (27)$$

Hence, to price option, we must obtain the first two risk-neutral moments (M_1, M_2) of the underlying asset at maturity. Then, we compute α and β using (27). Finally, we use the cumulative density function of the gamma distribution as $N(\cdot)$ in the Black-Scholes formula

In the next section, we compute the moments of the arithmetic mean and the ratios to price options on these assets with the methods previously described.

5.3 Computation of Moments

The following Lemma summarizes several properties of the stock price that will be used later:

Lemma 6 For $i, j = 1, 2, \dots, n$ and $k \in \mathbb{R}$, we have

$$E(S_i^k) = S_0^k \exp \left\{ k \left(r - q + \frac{k-1}{2} \sigma^2 \right) i \Delta t \right\} \quad (28)$$

$$E(S_i S_j^k) = E(S_i) E(S_j^k) \exp \{ k \sigma^2 \min\{i, j\} \Delta t \} \quad (29)$$

$$E(S_i S_j S_n^k) = E(S_i S_j) E(S_n^k) \exp \{ k \sigma^2 (i + j) \Delta t \} \quad (30)$$

Moreover, for $k \in \mathbb{R}$, we have

$$E(A_n S_n^k) = \frac{S_0}{n} E(S_n^k) h_1(r^*) \quad (31)$$

$$E(A_n^2 S_n^k) = \left(\frac{S_0}{n} \right)^2 E(S_n^k) [2f_1(r^* + \sigma^2)(h_1(2r^* + \sigma^2) - h_1(r^*)) - h_1(2r^* + \sigma^2)] \quad (32)$$

where

$$h_1(x) = \sum_{i=1}^n e^{xi\Delta t} = f_1(x) (e^{xn\Delta t} - 1), \quad x \neq 0, \quad h_1(0) = n \quad (33)$$

$$f_1(x) = \frac{e^{x\Delta t}}{e^{x\Delta t} - 1}, \quad x \neq 0 \quad (34)$$

$$r^* = r - q + k\sigma^2 \quad (35)$$

Proof: See Appendix.

The following Lemma gives the moments of A_n , S_n/A_n and A_n/S_n :

Lemma 7

1. The moments of the variable A_n are given by

$$E(A_n) = \frac{S_0}{n} h_1(r - q) \quad (36)$$

$$\text{Cov}(A_n, S_n) = \frac{S_0}{n} E(S_n) [h_1(r - q + \sigma^2) - h_1(r - q)] \quad (37)$$

$$\begin{aligned} V(A_n) &= \left(\frac{S_0}{n}\right)^2 [2f_1(r - q + \sigma^2)(h_1(2(r - q) + \sigma^2) - h_1(r - q)) \\ &\quad - h_1(2(r - q) + \sigma^2) - (h_1(r - q))^2] \end{aligned} \quad (38)$$

2. The moments of the variable S_n/A_n , $n \geq 2$ can be approximated by

$$\begin{aligned} E\left(\frac{S_n}{A_n}\right) &\simeq \frac{E(S_n)}{E(A_n)} - \frac{1}{(E(A_n))^2} \text{Cov}(A_n, S_n) + \frac{E(S_n)}{(E(A_n))^3} V(A_n) \\ V\left(\frac{S_n}{A_n}\right) &\simeq \left(\frac{E(S_n)}{E(A_n)}\right)^2 \left(\frac{V(S_n)}{(E(S_n))^2} + \frac{V(A_n)}{(E(A_n))^2} - 2\frac{\text{Cov}(A_n, S_n)}{E(S_n)E(A_n)}\right) \end{aligned}$$

with $E(A_n)$, $\text{Cov}(A_n, S_n)$ and $V(A_n)$ as given by (36)-(38).

3. The moments of the variable A_n/S_n , $n \geq 2$ are given by

$$E\left(\frac{A_n}{S_n}\right) = \frac{1}{n} \exp\{-n(r - q - \sigma^2)\Delta t\} h_1(r - q - \sigma^2) \quad (39)$$

$$\begin{aligned} V\left(\frac{A_n}{S_n}\right) &= \left(\frac{1}{n}\right)^2 \exp\{-n(2(r - q) - 3\sigma^2)\Delta t\} \\ &\quad \times [2f_1(r - q - \sigma^2)(h_1(2(r - q) - 3\sigma^2) - h_1(r - q - 2\sigma^2)) \\ &\quad - h_1(2(r - q) - 3\sigma^2) - \exp\{-n\sigma^2\Delta t\} h_1^2(r - q - \sigma^2)] \end{aligned} \quad (40)$$

with $h_1(\cdot)$ and $f_1(\cdot)$ as given by (33) and (34), respectively.

Proof: See Appendix.

Remark 1 Several particular cases can be highlighted:

1. If $r = q - \sigma^2$, the moments of the variable A_n are given by

$$\begin{aligned} E(A_n) &= \frac{S_0}{n} h_1(-\sigma^2) \\ \text{Cov}(A_n, S_n) &= \frac{S_0}{n} E(S_n)(n - h_1(-\sigma^2)) \\ V(A_n) &= 2 \left(\frac{S_0}{n} \right)^2 f_1(\sigma^2) e^{-(n+1)\sigma^2 \Delta t} \sum_{i=1}^n (\cosh(\sigma^2 i \Delta t) - 1) \end{aligned}$$

2. If $r = q + \sigma^2$, the moments of the variable A_n/S_n , $n \geq 2$ are given by

$$\begin{aligned} E\left(\frac{A_n}{S_n}\right) &= 1 \\ V\left(\frac{A_n}{S_n}\right) &= \left(\frac{1}{n}\right)^2 \left[(2f_1(\sigma^2) - 1) (e^{-\sigma^2 \Delta t} h_1(\sigma^2) - n) - n(n-1) \right] \end{aligned}$$

Proof: See Appendix.

The following Lemma gives the moments of A_T , S_T/A_T and A_T/S_T :

Lemma 8 For $k \in \mathbf{R}$, we have

$$E(A_T S_T^k) = \frac{S_0}{T} E(S_T^k) \Phi(r^*) \quad (41)$$

$$E(A_T^2 S_T^k) = 2 \left(\frac{S_0}{T} \right)^2 E(S_T^k) \frac{\Phi(2r^* + \sigma^2) - \Phi(r^*)}{r^* + \sigma^2} \quad (42)$$

with

$$\Phi(x) = \frac{\exp\{xT\} - 1}{x}, \quad x \neq 0, \quad \Phi(0) = T \quad (43)$$

and r^* as given by (35).

Proof: See Appendix.

Lemma 9

1. The moments of the variable A_T are given by

$$E(A_T) = \frac{S_0}{T} \Phi(r - q) \quad (44)$$

$$\text{Cov}(A_T, S_T) = \frac{S_0}{T} E(S_T) [\Phi(r - q + \sigma^2) - \Phi(r - q)] \quad (45)$$

$$V(A_T) = \left(\frac{S_0}{T}\right)^2 \left[2 \frac{\Phi(2(r - q) + \sigma^2) - \Phi(r - q)}{r - q + \sigma^2} - (\Phi(r - q))^2 \right] \quad (46)$$

2. The moments of the variable S_T/A_T can be approximated by

$$E\left(\frac{S_T}{A_T}\right) \simeq \frac{E(S_T)}{E(A_T)} - \frac{1}{(E(A_T))^2} \text{Cov}(A_T, S_T) + \frac{E(S_T)}{(E(A_T))^3} V(A_T)$$

$$V\left(\frac{S_T}{A_T}\right) \simeq \left(\frac{E(S_T)}{E(A_T)}\right)^2 \left(\frac{V(S_T)}{(E(S_T))^2} + \frac{V(A_T)}{(E(A_T))^2} - 2 \frac{\text{Cov}(A_T, S_T)}{E(S_T)E(A_T)} \right)$$

with $E(A_T)$, $\text{Cov}(A_T, S_T)$ and $V(A_T)$ as given by (44)-(46).

3. The moments of the variable A_T/S_T are given by

$$E\left(\frac{A_T}{S_T}\right) = \frac{1}{T} \Phi(\sigma^2 - (r - q)) \quad (47)$$

$$V\left(\frac{A_T}{S_T}\right) = \left(\frac{1}{T}\right)^2 \exp\{-(2(r - q) - 3\sigma^2)T\}$$

$$\times \left[2 \frac{\Phi(2(r - q) - 3\sigma^2) - \Phi(r - q - 2\sigma^2)}{r - q - \sigma^2} - \exp\{-\sigma^2 T\} \Phi^2(r - q - \sigma^2) \right] \quad (48)$$

Proof: See Appendix.

Remark 2 *Several particular cases can be highlighted:*

1. *If $r = q - \sigma^2$, the moments of the variable A_T are given by*

$$\begin{aligned} E(A_T) &= \frac{S_0}{T} \Phi(-\sigma^2) \\ \text{Cov}(A_T, S_T) &= \frac{S_0}{T} E(S_T) [T - \Phi(-\sigma^2)] \\ V(A_T) &= 2 \left(\frac{S_0}{T} \right)^2 \frac{e^{-\sigma^2 T}}{\sigma^4} (\sinh(\sigma^2 T) - \sigma^2 T) \end{aligned}$$

2. *If $r = q + \sigma^2$, the moments of the variable A_T/S_T are given by*

$$\begin{aligned} E\left(\frac{A_T}{S_T}\right) &= 1 \\ V\left(\frac{A_T}{S_T}\right) &= \left(\frac{1}{T}\right)^2 \left[2 \frac{\Phi(\sigma^2) - T}{\sigma^2} - T^2 \right] \end{aligned}$$

with $\Phi(\cdot)$ as given by (43).

Proof: See Appendix.

Tables 3 and 4 show arithmetic call and put option prices (multiplied by 100) for different monitoring dates. The interest rate is 10 % and the stock dividend yield is 3 %. We price options on $A_n, S_n/A_n$ and A_n/S_n with three methods: Monte Carlo simulation¹⁴, Wilkinson approximation, and gamma distribution.

[Insert Tables 3 and 4 about here]

In the tables, we see that derivative prices with the three methods are very close. For example, in Table 3 we have that when $\sigma = 0.20, T = 0.5, K = 0.8$,

¹⁴We use 10,000 simulations and antithetic variables to reduce standard errors.

and $n = 1,000$, the values of call options on A_n/S_n are 18.319, 18.324, and 18.321, respectively. Thus, Edgeworth expansions do not seem to be needed.

To price options on S_n/A_n with both the Wilkinson approximation and the gamma distribution, we have computed its moments using the approximation of Mood *et al* (1974). In the tables we see that those prices are very similar to the ones obtained with Monte Carlo, so that the approximations seem to work pretty well. For example, in Table 3 we see that when $\sigma = 0.2, T = 0.5, K = 0.8$, and $n = 1,000$, the values of call options using the Wilkinson approximation and the gamma distribution are 20.375 and 20.374, respectively, while the value obtained with Monte Carlo simulation is 20.377.

When the average is computed in continuous time (number of monitoring dates = ∞) we cannot use Monte Carlo simulation. However, as mentioned before, using 1,000 monitoring dates produces option prices very similar to those using continuous average. In Table 4 we see that the values of put options on S_n/A_n using the Wilkinson approximation and the gamma distribution are 8.863 and 8.887, respectively, for both 1,000 and ∞ monitoring dates.

To understand better why the three methods produce very similar results, we plot the risk-neutral probability density function of the arithmetic stock price average in Figure 6.

[Insert Figure 6 about here]

The parameter values are: $r = 0.1, q = 0, \sigma = 0.2, T = 1, S_0 = 100$ and $n = \infty$. The expected value of the average price is 105.17, and the variance 152.74. For $n = \infty$ the true density function is reciprocal gamma, with

parameters $\alpha = 74.42$ and $\beta = 1.29\text{E-}4$. This function is approximated with a lognormal distribution with the same moments. The density function is also estimated with Monte Carlo simulation, using a set of 50 runs of 10,000 paths with 1,000 time steps. We see that, for the parameter values used, the density functions are remarkably similar, hence the price of options on arithmetic stock prices must be close.

6 Conclusions

Australian Asian options are options on the ratio of the stock price to its average or viceversa. They show up in variable purchase options, recently studied by Handley (2000).

If the stock price follows a geometric Brownian motion and the average is defined on geometric basis, these ratios also follow a geometric Brownian motion. Thus, we are able to obtain closed-form expressions for the price of the options. However, when the average is defined on arithmetic basis, the risk-neutral distributions of these ratios at maturity are unknown. Hence, to price the options we use a particular case of Edgeworth expansion (known as Wilkinson approximation) as well as a gamma approximation (following Milevsky and Posner (1998)). We compare the results with those obtained with Monte Carlo simulations, and we find that option prices are very similar in the three cases.

Appendix

Proof of Lemma 1

1. For proving expressions (2)-(3), see Johnson and Kotz (1970), p. 115.

As $X^{-1} = e^{-Y}$, application of (2)-(3) gives

$$\begin{aligned} E(X^{-1}) &= \exp\left\{-m + \frac{1}{2}s^2\right\} = e^{-2m} E(X) \\ V(X^{-1}) &= [E(X^{-1})]^2 [e^{s^2} - 1] \end{aligned}$$

2. Consider the variable

$$\tilde{X} = \begin{cases} X & \text{if } X \geq K \\ 0 & \text{if } X < K \end{cases}, \quad K \in \mathbb{R}^+$$

Using (1), the expectation of this variable is

$$E(\tilde{X}) = \int_K^\infty \frac{1}{s\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln x - m}{s}\right)^2\right\} dx$$

Defining the variable

$$y = \frac{\ln x - m}{s},$$

a little algebra leads to

$$E(\tilde{X}) = E(X)N(s - D)$$

where $N(\cdot)$ denotes the distribution function of a standard normal variable and $D = \frac{\ln K - m}{s}$.

Proof of Lemma 3

The expressions for the mean and variance follow from straightforward application of (2)-(3) in Lemma 1 to S_t as given by (12).

To obtain $\text{Cov}(Z_t, Z_s)$, we assume $s < t$. We first compute $E(Z_t Z_s)$:

$$\begin{aligned} E(Z_t Z_s) &= Z_0^2 E \left(\exp \left\{ \left(\alpha_Z - \frac{1}{2} \sigma_Z^2 \right) (t+s) + \sigma_Z (W_t + W_s) \right\} \right) \\ &= Z_0^2 \exp \left\{ \left(\alpha_Z - \frac{1}{2} \sigma_Z^2 \right) (t+s) + \frac{1}{2} \sigma_Z^2 V(W_t + W_s) \right\} \\ &= Z_0^2 \exp \left\{ \left(\alpha_Z - \frac{1}{2} \sigma_Z^2 \right) (t+s) + \frac{1}{2} \sigma_Z^2 (t+s+2s) \right\} \\ &= Z_0^2 \exp \{ \alpha_Z (t+s) + \sigma_Z^2 s \} \end{aligned}$$

where we have used (2) in Lemma 1. Then,

$$\begin{aligned} \text{Cov}(Z_t, Z_s) &= E(Z_t Z_s) - E(Z_t)E(Z_s) = Z_0^2 e^{\alpha_Z(t+s) + \sigma_Z^2 s} - (Z_0 e^{\alpha_Z t}) (Z_0 e^{\alpha_Z s}) \\ &= Z_0^2 e^{\alpha_Z(t+s)} [e^{\sigma_Z^2 s} - 1] \end{aligned}$$

Proof of Proposition 1

We split the option payoff in two components:

1. The “contingent exercise payment”, a claim with payoff

$$C_1(Z, T, T, K) = -K I_{\{Z_T \geq K\}} = \begin{cases} -K & \text{if } Z_T \geq K \\ 0 & \text{if } Z_T < K \end{cases}$$

2. The “contingent receipt of the stock”, a claim with payoff

$$C_2(Z, T, T, K) = Z_T I_{\{Z_T \geq K\}} = \begin{cases} Z_T & \text{if } Z_T \geq K \\ 0 & \text{if } Z_T < K \end{cases}$$

We will compute the two components of the option:

$$\begin{aligned} C_1(Z, 0, T, K) &= E \left[e^{-rT} C_1(Z, T, T, K) \mid F_t \right] = -K e^{-rT} P(Z_T \geq K) \\ C_2(Z, 0, T, K) &= E \left[e^{-rT} C_2(Z, T, T, K) \mid F_t \right] = e^{-rT} E[Z_T \mid Z_T \geq K] \end{aligned} \quad (50)$$

Equation (8) implies

$$\frac{\ln \left(e^{-\alpha_Z T} E(Z_T) / Z_T \right) + \left(\alpha_Z - \frac{1}{2} \sigma_Z^2 \right) T}{\sigma_Z \sqrt{T}} \sim N(0, 1)$$

and, then, it is verified that

$$P(Z_T \geq K) = N(d_2) \quad (51)$$

where

$$d_2 = \frac{\ln(e^{-\alpha_Z T} E(Z_T) / K) + \left(\alpha_Z - \frac{1}{2} \sigma_Z^2 \right) T}{\sigma_Z \sqrt{T}}$$

Part 2 in Lemma 1 and a little algebra leads to

$$E[Z_T | Z_T \geq K] = E(Z_T)N(d_1) \quad (52)$$

where $d_1 = d_2 + \sigma_Z\sqrt{T}$.

Including (51)-(52) into (49)-(50), we obtain the final expression for the option price.

Proof of Lemma 4

For $n \geq 2$, we have

$$\begin{aligned}
 V\left(\sum_{i=1}^n W_{t_i}\right) &= V\left(\sum_{i=1}^{n-1} W_{t_i} + W_{t_n}\right) \\
 &= V\left(\sum_{i=1}^{n-1} W_{t_i}\right) + t_n + 2\sum_{i=1}^{n-1} \text{Cov}(W_{t_i}, W_{t_n}) \\
 &= V\left(\sum_{i=1}^{n-1} W_{t_i}\right) + n\Delta t + 2\sum_{i=1}^{n-1} i\Delta t \\
 &= V\left(\sum_{i=1}^{n-1} W_{t_i}\right) + n^2\Delta t
 \end{aligned}$$

By induction, we get

$$V\left(\sum_{i=1}^n W_{t_i}\right) = \sum_{i=1}^n i^2 \Delta t = \frac{(n+1)\left(n+\frac{1}{2}\right)n}{3} \Delta t$$

$$\begin{aligned}
 &V\left(n W_{t_n} - \sum_{i=1}^n W_{t_i}\right) \\
 &= V(n W_{t_n}) + V\left(\sum_{i=1}^n W_{t_i}\right) - 2\text{Cov}\left(n W_{t_n}, \sum_{i=1}^n W_{t_i}\right) \\
 &= n^2 V(W_{t_n}) + \frac{(n+1)\left(n+\frac{1}{2}\right)n}{3} \Delta t - 2n \sum_{i=1}^n \text{Cov}(W_{t_n}, W_{t_i}) \\
 &= n^2 t_n + \frac{(n+1)\left(n+\frac{1}{2}\right)n}{3} \Delta t - 2n \sum_{i=1}^n t_i \\
 &= \left[n^2 + \frac{(n+1)\left(n+\frac{1}{2}\right)}{3} - 2 \sum_{i=1}^n i \right] n \Delta t \\
 &= \frac{(n-1)\left(n-\frac{1}{2}\right)n}{3} \Delta t
 \end{aligned}$$

Proof of Proposition 2

These formulae are consequence of Proposition 1 and the moments of Z_n that we get now:

- Departing from (13) and applying (2)-(3) in Lemma 1 and Lemma 4, we obtain the moments of G_n .
- The same procedure applied to (14) provides the moments of S_n/G_n .
- Comparing (14) and (15) and applying (4)-(5) in Lemma 1, we obtain the moments of G_n/S_n .

Proof of Lemma 5

$$\begin{aligned}
& V\left(\int_0^T W_t dt\right) \\
&= E\left(\left(\int_0^T W_t dt\right)^2\right) - \left(E\left(\int_0^T W_t dt\right)\right)^2 \\
&= E\left(\left(\int_0^T W_t dt\right)^2\right) = E\left(\int_0^T W_t dt \int_0^T W_s ds\right) \\
&= E\left(\int_0^T W_t \left(\int_0^T W_s ds\right) dt\right) = E\left(\int_0^T \int_0^T W_t W_s ds dt\right) \\
&= \int_0^T \int_0^T E(W_t W_s) ds dt = \int_0^T \int_0^T \text{Cov}(W_t W_s) ds dt \\
&= \int_0^T \int_0^t \text{Cov}(W_t W_s) ds dt + \int_0^T \int_t^T \text{Cov}(W_t W_s) ds dt \\
&= \int_0^T \int_0^t s ds dt + \int_0^T \int_t^T t ds dt \\
&= \frac{T^3}{3}
\end{aligned}$$

$$\begin{aligned}
& V\left(T W_T - \int_0^T W_t dt\right) \\
&= V(T W_T) + V\left(\int_0^T W_t dt\right) - 2\text{Cov}\left(T W_T, \int_0^T W_t dt\right) \\
&= T^2 T + \frac{T^3}{3} - 2T \int_0^T \text{Cov}(W_T, W_t) dt \\
&= 4\frac{T^3}{3} - 2T \int_0^T t dt \\
&= \frac{T^3}{3}
\end{aligned}$$

Proof of Proposition 3

These formulae are consequence of Proposition 1 and the moments of Z_T that we obtain now:

- Departing from (16) and applying (2)-(3) in Lemma 1 and Lemma 5, we obtain the moments of G_T .
- The same procedure applied to (17) provides the moments of S_T/G_T .
- Comparing (17) and (18) and applying (4)-(5) in Lemma 1, we obtain the moments of G_T/S_T .

Proof of Lemma 6

Using (12) and a little algebra, we have, for $a, b, k \in \mathbb{R}$,

$$\begin{aligned}
& E(S_i^a S_j^b S_n^k) \\
&= S_0^{a+b+k} E\left(\exp\left\{\left(r - q - \frac{1}{2}\sigma^2\right)(at_i + bt_j + kt_n) + \sigma(aW_{t_i} + bW_{t_j} + kW_{t_n})\right\}\right) \\
&= S_0^a \exp\left\{\left(r - q + \frac{a-1}{2}\sigma^2\right)ai\Delta t\right\} S_0^b \exp\left\{\left(r - q + \frac{b-1}{2}\sigma^2\right)bj\Delta t\right\} \\
&\quad \times S_0^k \exp\left\{\left(r - q + \frac{k-1}{2}\sigma^2\right)kn\Delta t\right\} \exp\{ab\sigma^2 \min\{i, j\}\Delta t\} \\
&\quad \times \exp\{k\sigma^2(ai + bj)\Delta t\} \tag{53}
\end{aligned}$$

Several particular cases are the following:

$$\begin{aligned}
b = k = 0 &\Rightarrow E(S_i^a) = S_0^a \exp\left\{\left(r - q + \frac{a-1}{2}\sigma^2\right)ai\Delta t\right\} \\
a = k = 0 &\Rightarrow E(S_j^b) = S_0^b \exp\left\{\left(r - q + \frac{b-1}{2}\sigma^2\right)bj\Delta t\right\} \\
a = b = 0 &\Rightarrow E(S_n^k) = S_0^k \exp\left\{\left(r - q + \frac{k-1}{2}\sigma^2\right)kn\Delta t\right\}
\end{aligned}$$

Plugging these expressions into (53), we get

$$E(S_i^a S_j^b S_n^k) = E(S_i^a)E(S_j^b)E(S_n^k) \exp\{ab\sigma^2 \min\{i, j\}\Delta t\} \exp\{k\sigma^2(ai + bj)\Delta t\}$$

For $k = 0$, we get

$$E(S_i^a S_j^b) = E(S_i^a)E(S_j^b) \exp\{ab\sigma^2 \min\{i, j\}\Delta t\} \tag{54}$$

and, then,

$$E(S_i^a S_j^b S_n^k) = E(S_i^a S_j^b)E(S_n^k) \exp\{k\sigma^2(ai + bj)\Delta t\} \tag{55}$$

Using (54) with $a = 1, b = k$ and (55) with $a = b = 1$, we obtain $E(S_i S_j^k)$ and $E(S_i S_j S_n^k)$, respectively.

- Mean of $A_n S_n^k$

$$E(A_n S_n^k) = E\left(\frac{1}{n} \sum_{i=1}^n S_i S_n^k\right) = \frac{1}{n} \sum_{i=1}^n E(S_i S_n^k)$$

Using (29) with $j = n$, we have

$$\begin{aligned} E(A_n S_n^k) &= \frac{1}{n} \sum_{i=1}^n E(S_i) E(S_n^k) e^{k\sigma^2 i \Delta t} = \frac{1}{n} E(S_n^k) \sum_{i=1}^n S_0 e^{(r-q+k\sigma^2) i \Delta t} \\ &= \frac{S_0}{n} E(S_n^k) h_1(r^*) \end{aligned}$$

with $h_1(\cdot)$ as given by (33) and $r^* = r - q + k\sigma^2$.

- Mean of $A_n^2 S_n^k$

$$\begin{aligned} E(A_n^2 S_n^k) &= E\left(\left(\frac{1}{n} \sum_{i=1}^n S_i\right)^2 S_n^k\right) \\ &= \left(\frac{1}{n}\right)^2 E\left(\sum_{i,j=1}^n S_i S_j S_n^k\right) = \left(\frac{1}{n}\right)^2 \sum_{i,j=1}^n E(S_i S_j S_n^k) \\ &= \left(\frac{1}{n}\right)^2 E(S_n^k) \sum_{i,j=1}^n E(S_i S_j) e^{k\sigma^2(i+j)\Delta t} \end{aligned} \quad (56)$$

where the last equation results from (30).

We define

$$z_n^* = \sum_{i,j=1}^n E(S_i S_j) e^{k\sigma^2(i+j)\Delta t}$$

Then, we have

$$\begin{aligned} z_n^* &= \sum_{i,j=1}^{n-1} E(S_i S_j) e^{k\sigma^2(i+j)\Delta t} + 2 \sum_{i=1}^n E(S_i S_n) e^{k\sigma^2(i+n)\Delta t} - E(S_n S_n) e^{2k\sigma^2 n \Delta t} \\ &= z_{n-1}^* + 2 \sum_{i=1}^n E(S_i) E(S_n) e^{\sigma^2 i \Delta t} e^{k\sigma^2(i+n)\Delta t} - E(S_n^2) e^{2k\sigma^2 n \Delta t} \end{aligned}$$

After some algebra, we get the recurrence law

$$z_n^* = z_{n-1}^* + S_0^2 \left[2f_1(r^* + \sigma^2) \left(e^{(2r^* + \sigma^2)n\Delta t} - e^{r^*n\Delta t} \right) - e^{(2r^* + \sigma^2)n\Delta t} \right] \quad (57)$$

If we compute z_1^* either by its definition or using (57) and compare both results, we obtain $z_0^* = 0$.

Using (57) for different values of n , we obtain

$$z_n^* = S_0^2 \left[2f_1(r^* + \sigma^2)(h_1(2r^* + \sigma^2) - h_1(r^*)) - h_1(2r^* + \sigma^2) \right]$$

Plugging this expression into (56), we have

$$E(A_n^2 S_n^k) = \left(\frac{S_0}{n} \right)^2 E(S_n^k) [2f_1(r^* + \sigma^2)(h_1(2r^* + \sigma^2) - h_1(r^*)) - h_1(2r^* + \sigma^2)]$$

Proof of Lemma 7

1. Moments of the arithmetic average A_n

(a) **Mean of A_n :**

Apply (31) with $k = 0$.

(b) **Covariance of A_n with S_n :**

Apply (31) for $k = 1$ and (36).

(c) **Variance of A_n :**

Apply (32) for $k = 0$ and (36).

2. Moments of the variable S_n/A_n :

Apply part 2 in Lemma 2 with $X = S_n$, $Y = A_n$.

3. Moments of the variable A_n/S_n

(a) **Mean of A_n/S_n :**

Apply (28) for $i = n$, $k = -1$ and (31) for $k = -1$.

(b) **Variance of A_n/S_n :**

Apply (28) for $i = n$, $k = -2$, (32) for $k = -2$ and (39).

Proof of Remark 1

1. Replace $r = q - \sigma^2$ into (36)-(37) to obtain $E(A_n)$ and $\text{Cov}(A_n, S_n)$.

To compute $V(A_n)$, we will need the following relationships, satisfied by the functions $h_1(\cdot)$ and $f_1(\cdot)$ (see (33)-(34)):

$$h_1(-a) = \exp\{-(n+1)a\Delta t\}h_1(a) \quad (58)$$

$$f_1(-a) = -\exp\{-a\Delta t\}f_1(a) \quad (59)$$

$$1 + h_1(a) = f_1(-a) \left[1 - e^{(n+1)a\Delta t}\right] \quad (60)$$

$$f_1(b) = \exp\{(b-a)\Delta t\} \frac{e^{a\Delta t} - 1}{e^{b\Delta t} - 1} f_1(a) \quad (61)$$

Looking at (38), we need to compute

$$f_1(r - q + \sigma^2)[h_1(2(r - q) + \sigma^2) - h_1(r - q)]$$

Defining $x = r - q + \sigma^2$ and using (61), this expression becomes

$$f_1(x)f_1(x - \sigma^2) \left[e^{x\Delta t} \frac{e^{(x - \sigma^2)\Delta t} - 1}{e^{(2x - \sigma^2)\Delta t} - 1} (e^{(2x - \sigma^2)n\Delta t} - 1) - (e^{(2x - \sigma^2)\Delta t} - 1) \right]$$

Taking limits when $x \rightarrow 0$ and applying (59) and some algebra, we obtain

$$f_1(\sigma^2) \left[h_1(-\sigma^2) - ne^{-(n+1)\sigma^2\Delta t} \right]$$

Replacing this result into (38) and using (58) and (60), we obtain

$$V(A_n) = \left(\frac{S_0}{n}\right)^2 f_1(\sigma^2) e^{-(n+1)\sigma^2\Delta t} [h_1(\sigma^2) - 2n + h_1(-\sigma^2)]$$

It can be seen that, as expected, this variance is positive since

$$h_1(\sigma^2) - 2n + h_1(-\sigma^2) = 2 \sum_{i=1}^n (\cosh(\sigma^2 i \Delta t) - 1) > 0$$

2. Replace $r = q + \sigma^2$ into (39) and use $h_1(0) = n$ to obtain $E(A_n/S_n)$.

To compute $V(A_n/S_n)$, looking at (40), we need to compute

$$f_1(r - q - \sigma^2)[h_1(2(r - q) - 3\sigma^2) - h_1(r - q - 2\sigma^2)]$$

Defining $x = r - q - \sigma^2$ and applying the same procedure as in part 2 of this remark, this expression becomes

$$f_1(\sigma^2) [h_1(-\sigma^2) - ne^{-(n+1)\sigma^2\Delta t}]$$

Replacing this result into (40) and using (58), we obtain

$$V\left(\frac{A_n}{S_n}\right) = \left(\frac{1}{n}\right)^2 [(2f_1(\sigma^2) - 1)(e^{-\sigma^2\Delta t}h_1(\sigma^2) - n) - n(n - 1)]$$

Proof of Lemma 8

- Mean of $A_T S_T^k$

$$E(A_T S_T^k) = E\left(\frac{1}{T} \int_0^T S_t dt S_T^k\right) = \frac{1}{T} \int_0^T E(S_t S_T^k) dt$$

Using (29) with $j = n$, we have

$$\begin{aligned} E(A_T S_T^k) &= \frac{1}{T} \int_0^T E(S_t) E(S_T^k) \exp\{k\sigma^2 t\} dt \\ &= \frac{1}{T} E(S_T^k) \int_0^T S_0 \exp\{(r - q)t\} \exp\{k\sigma^2 t\} dt \\ &= \frac{S_0}{T} E(S_T^k) \Phi(r^*) \end{aligned}$$

with $\Phi(\cdot)$ as given by (43) and $r^* = r - q + k\sigma^2$.

- Mean of $A_T^2 S_T^k$

Use (24) and apply (29)-(30).

Proof of Lemma 9

1. Moments of the arithmetic average A_T

(a) **Mean of A_T :**

Apply (41) with $k = 0$.

(b) **Covariance of A_T with S_T :**

Apply (41) for $k = 1$ and (44).

(c) **Variance of A_T :**

Apply (42) for $k = 0$ and (44).

2. Moments of the variable S_T/A_T

Apply part 2 in Lemma 2 with $X = S_T$, $Y = A_T$.

3. Moments of the variable A_T/S_T

(a) **Mean of A_T/S_T**

Apply (28) for $i = n, k = -1$ and (41) for $k = -1$.

(b) **Variance of A_T/S_T**

Apply (28) for $i = n, k = -2$, (42) for $k = -2$ and (47).

Proof of Remark 2

1. Replace $r = q - \sigma^2$ into (44)-(45) to obtain $E(A_T)$ and $\text{Cov}(A_T, S_T)$.

To compute $V(A_T)$, looking at (46), we need to compute

$$\lim_{r \rightarrow q - \sigma^2} \frac{\Phi(2(r - q) + \sigma^2) - \Phi(r - q)}{r - q + \sigma^2}$$

that is easily computed applying the L'Hôpital's rule. Plugging this result into (46) and using the relationship $\Phi(-x) = e^{-xT} \Phi(x)$, we obtain

$$V(A_T) = \left(\frac{S_0}{T}\right)^2 \frac{e^{-\sigma^2 T}}{\sigma^2} [\Phi(\sigma^2) - 2T + \Phi(-\sigma^2)]$$

It can be seen that, as expected, this variance is positive since

$$\Phi(\sigma^2) - 2T + \Phi(-\sigma^2) = \frac{2}{\sigma^2} [\sinh(\sigma^2 T) - \sigma^2 T] > 0$$

2. Replace $r = q + \sigma^2$ into (47) and use $\Phi(0) = T$ to obtain $E(A_T/S_T)$.

To compute $V(A_T/S_T)$, looking at (48), we need to compute

$$\lim_{r \rightarrow q + \sigma^2} \frac{\Phi(2(r - q) - 3\sigma^2) - \Phi(r - q - 2\sigma^2)}{r - q - \sigma^2}$$

Applying the L'Hôpital's rule and plugging the result into (48), we obtain

$$V\left(\frac{A_T}{S_T}\right) = \left(\frac{1}{T}\right)^2 \left[2 \frac{\Phi(\sigma^2) - T}{\sigma^2} - T^2 \right]$$

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Table 1. Geometric Call option prices.

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
20	0.5	0.8	S_n	22.576	22.576	22.576	22.576	22.576
			G_n	22.576	20.696	20.561	20.548	20.546
			S_n/G_n	19.025	20.400	20.531	20.545	20.546
			G_n/S_n	19.025	18.133	18.161	18.166	18.166
20	1	0.8	S_n	25.187	25.187	25.187	25.187	25.187
			G_n	25.187	21.361	21.084	21.056	21.053
			S_n/G_n	18.097	20.756	21.023	21.050	21.053
			G_n/S_n	18.097	16.491	16.580	16.593	16.594
40	0.5	0.8	S_n	24.801	24.801	24.801	24.801	24.801
			G_n	24.801	20.766	20.558	20.538	20.536
			S_n/G_n	19.025	20.332	20.514	20.534	20.536
			G_n/S_n	19.025	20.266	20.908	20.979	20.987
20	0.5	1.1	S_n	3.175	3.175	3.175	3.175	3.175
			G_n	3.175	0.905	0.737	0.721	0.719
			S_n/G_n	0	0.549	0.701	0.717	0.719
			G_n/S_n	0	0.282	0.374	0.384	0.385

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on S_n and G_n we take $S_0 = 1$. For options on G_n , derivative prices can be computed with the Merton's (1973) formula when $n = 1$, and with the Kemna and Vorst's (1990) formula when $n = \infty$.

Table 2. Geometric Put option prices.

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
20	0.5	1.0	S_n	3.930	3.930	3.930	3.930	3.930
			G_n	3.930	2.591	2.439	2.423	2.422
			S_n/G_n	0	2.250	2.405	2.420	2.422
			G_n/S_n	0	3.321	3.516	3.535	3.537
20	1	1.0	S_n	4.639	4.639	4.639	4.639	4.639
			G_n	4.639	3.135	2.955	2.937	2.935
			S_n/G_n	0	2.732	2.915	2.933	2.935
			G_n/S_n	0	4.724	4.976	5.000	5.002
40	0.5	1.0	S_n	9.277	9.277	9.277	9.277	9.277
			G_n	9.277	6.149	5.763	5.724	5.719
			S_n/G_n	0	5.283	5.676	5.715	5.719
			G_n/S_n	0	5.314	5.493	5.508	5.510
20	0.5	1.1	S_n	9.300	9.300	9.300	9.300	9.300
			G_n	9.300	8.753	8.712	8.714	8.713
			S_n/G_n	9.512	8.688	8.710	8.713	8.713
			G_n/S_n	9.512	10.691	10.759	10.765	10.766

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on S_n and G_n we take $S_0 = 1$. For options on G_n , derivative prices can be computed with the Merton's (1973) formula when $n = 1$, and with the Kemna and Vorst's (1990) formula when $n = \infty$.

Table 3. Arithmetic Call option prices.

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
20	0.5	0.8	A_n MC	22.551	20.737	20.731	20.723	-
			A_n W	22.576	20.885	20.729	20.714	20.712
			A_n GD	22.535	20.883	20.728	20.711	20.711
			S_n/A_n MC	19.025	20.329	20.381	20.377	-
			S_n/A_n W	19.025	20.209	20.360	20.375	20.377
			S_n/A_n GD	19.025	20.208	20.358	20.374	20.375
			A_n/S_n MC	19.025	18.332	18.317	18.319	-
			A_n/S_n W	19.025	18.389	18.330	18.324	18.324
			A_n/S_n GD	19.025	18.388	18.327	18.321	18.321
20	1	0.8	A_n MC	25.089	21.431	21.366	21.387	-
			A_n W	25.187	21.736	21.418	21.387	21.384
			A_n GD	25.059	21.716	21.404	21.373	21.370
			S_n/A_n MC	18.097	20.642	20.730	20.766	-
			S_n/A_n W	18.097	20.376	20.682	20.713	20.717
			S_n/A_n GD	18.097	20.366	20.667	20.698	20.701
			A_n/S_n MC	18.097	16.841	16.884	16.899	-
			A_n/S_n W	18.097	16.964	16.887	16.880	16.883
			A_n/S_n GD	18.097	16.947	16.862	16.855	16.857

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on A_n we take $S_0 = 1$. MC, W and GD refer to Monte Carlo simulation, Wilkinson approximation and gamma distribution, respectively.

Table 3. Arithmetic Call option prices (cont.).

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
40	0.5	0.8	A_n MC	24.687	21.141	21.111	21.206	-
			A_n W	24.801	21.468	21.188	21.161	21.158
			A_n GD	24.400	21.358	21.099	21.074	21.071
			S_n/A_n MC	19.025	19.853	20.011	19.987	-
			S_n/A_n W	19.025	19.641	19.918	19.947	19.951
			S_n/A_n GD	19.025	19.562	19.819	19.846	19.849
			A_n/S_n MC	19.025	21.432	21.620	21.535	-
			A_n/S_n W	19.025	21.276	21.569	21.599	21.620
			A_n/S_n GD	19.025	21.213	21.489	21.517	21.535
20	0.5	1.1	A_n MC	3.141	0.802	0.756	0.757	-
			A_n W	3.176	0.933	0.778	0.761	0.759
			A_n GD	3.198	0.980	0.802	0.785	0.782
			S_n/A_n MC	0	0.593	0.612	0.682	-
			S_n/A_n W	0	0.528	0.681	0.698	0.699
			S_n/A_n GD	0	0.549	0.705	0.721	0.722
			A_n/S_n MC	0	0.389	0.404	0.413	-
			A_n/S_n W	0	0.302	0.387	0.396	0.401
			A_n/S_n GD	0	0.320	0.409	0.419	0.424

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on A_n we take $S_0 = 1$. MC, W and GD refer to Monte Carlo simulation, Wilkinson approximation and gamma distribution, respectively.

Table 4. Arithmetic Put option prices.

Parameters			Asset	Number of monitoring dates (n)				
σ (%)	T	K	Z_n	1	10	100	1,000	∞
20	0.5	1.0	A_n MC	3.930	2.331	2.417	2.387	-
			A_n W	3.930	2.530	2.387	2.372	2.371
			A_n GD	3.868	2.511	2.371	2.357	2.356
			S_n/A_n MC	0	2.365	2.477	2.512	-
			S_n/A_n W	0	2.342	2.490	2.505	2.506
			S_n/A_n GD	0	2.330	2.475	2.491	2.494
			A_n/S_n MC	0	3.328	3.440	3.408	-
			A_n/S_n W	0	3.176	3.417	3.440	3.451
			A_n/S_n GD	0	3.175	3.415	3.440	3.449
20	1	1.0	A_n MC	4.639	2.821	2.771	2.872	-
			A_n W	4.639	3.038	2.875	2.859	2.857
			A_n GD	4.479	2.991	2.834	2.819	2.817
			S_n/A_n MC	0	3.043	2.994	3.055	-
			S_n/A_n W	0	2.913	3.086	3.103	3.105
			S_n/A_n GD	0	2.884	3.051	3.068	3.069
			A_n/S_n MC	0	4.736	4.854	4.817	-
			A_n/S_n W	0	4.436	4.775	4.809	4.833
			A_n/S_n GD	0	4.433	4.770	4.803	4.827

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on A_n we take $S_0 = 1$. MC, W and GD refer to Monte Carlo simulation, Wilkinson approximation and gamma distribution, respectively.

Table 4. Arithmetic Put option prices (cont.).

Parameters			Asset	Number of monitoring dates (n)				
σ (%)	T	K	Z_n	1	10	100	1,000	∞
40	0.5	1.0	A_n MC	9.277	5.465	5.402	5.554	-
			A_n W	9.277	5.874	5.525	5.490	5.487
			A_n GD	8.971	5.792	5.457	5.423	5.419
			S_n/A_n MC	0	5.855	6.000	5.988	-
			S_n/A_n W	0	5.837	6.202	6.238	6.242
			S_n/A_n GD	0	5.794	6.147	6.182	6.186
			A_n/S_n MC	0	5.078	5.202	5.201	-
			A_n/S_n W	0	4.880	5.193	5.224	5.290
			A_n/S_n GD	0	4.822	5.124	5.154	5.217
20	0.5	1.1	A_n MC	9.300	8.571	8.581	8.576	-
			A_n W	9.300	8.613	8.589	8.588	8.588
			A_n GD	9.322	8.639	8.613	8.611	8.611
			S_n/A_n MC	9.512	8.838	8.827	8.829	-
			S_n/A_n W	9.512	8.858	8.862	8.863	8.863
			S_n/A_n GD	9.512	8.879	8.886	8.887	8.887
			A_n/S_n MC	9.512	10.586	10.626	10.617	-
			A_n/S_n W	9.512	10.453	10.603	10.618	10.623
			A_n/S_n GD	9.512	10.472	10.625	10.640	10.646

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on A_n we take $S_0 = 1$. MC, W and GD refer to Monte Carlo simulation, Wilkinson approximation and gamma distribution, respectively.

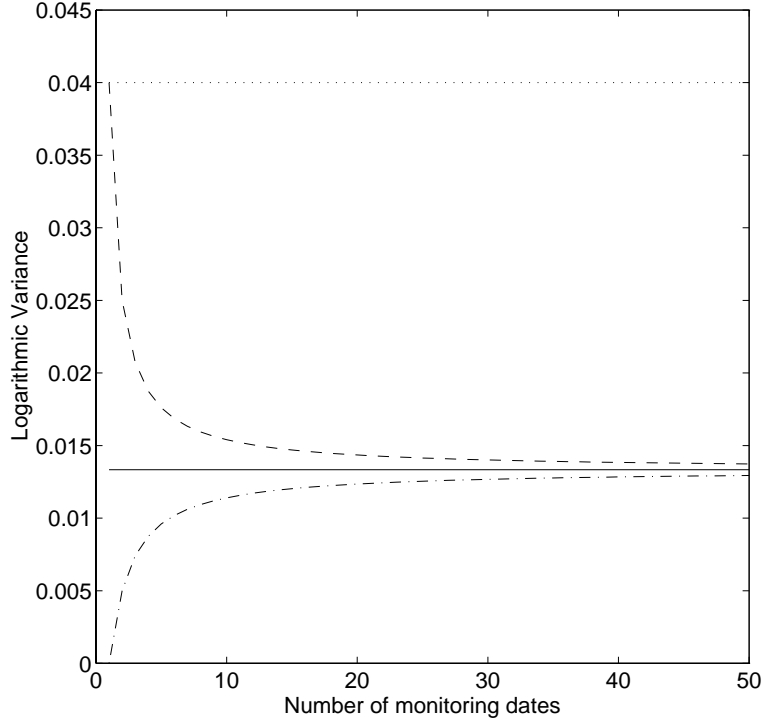


Figure 1: **Plot of the logarithmic variance ($\sigma_Z^2 T$) as function of n .** The parameter values are $\sigma = 0.2$ and $T = 1$. The figure depicts the logarithmic variance of the stock price (dotted line), G_n (dashed line), G_T (solid line), S_n/G_n (dotted-dashed line), and G_n/S_n . These values are given by the following table:

Z_n	$\sigma_Z^2 T$
S_n	$\sigma^2 T$
G_n	$\frac{(n+1)(n+\frac{1}{2})}{3n^2} \sigma^2 T$
S_n/G_n	$\frac{(n-1)(n-\frac{1}{2})}{3n^2} \sigma^2 T$
G_n/S_n	$\frac{(n-1)(n-\frac{1}{2})}{3n^2} \sigma^2 T$

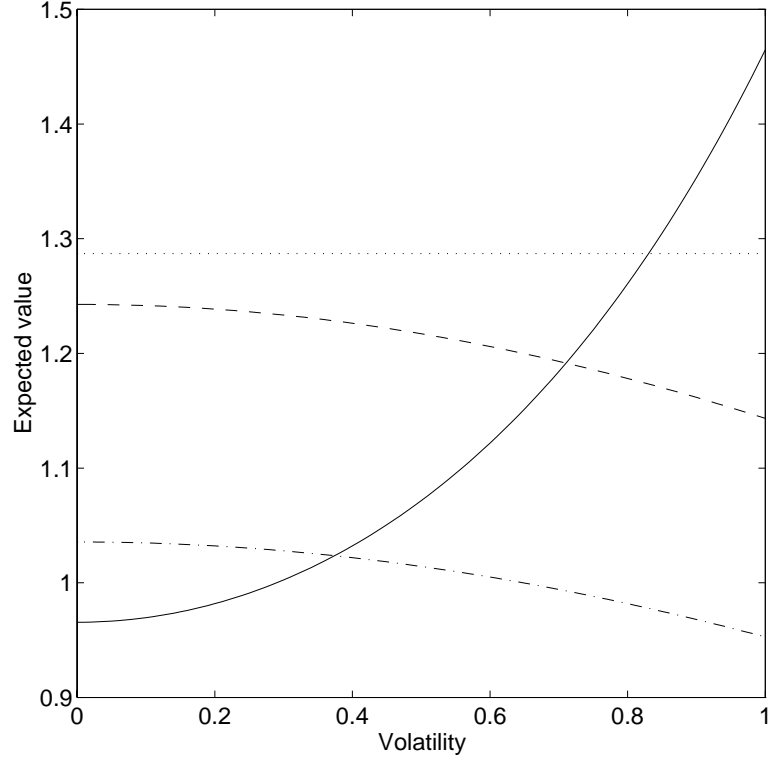


Figure 2: **Plot of the expected value $E(Z_T)$ as function of σ^2 .** The parameter values are $r = 10\%$, $q = 3\%$, $\sigma = 0.2$ and $T = 1$. We assume $S_0 = 1.2$. The figure depicts the expected values of the stock price (dotted line), G_T (dashed line), S_T/G_T (dotted-dashed line), and G_T/S_T (solid line). These values are given by the following table:

Z_T	$E(Z_T)$
S_T	$S_0 \exp\{(r - q)T\}$
G_T	$S_0 \exp\left\{\frac{1}{2}\left(r - q - \frac{1}{6}\sigma^2\right)T\right\}$
S_T/G_T	$\exp\left\{\frac{1}{2}\left(r - q - \frac{1}{6}\sigma^2\right)T\right\}$
G_T/S_T	$\exp\left\{-\frac{1}{2}\left(r - q - \frac{5}{6}\sigma^2\right)T\right\}$

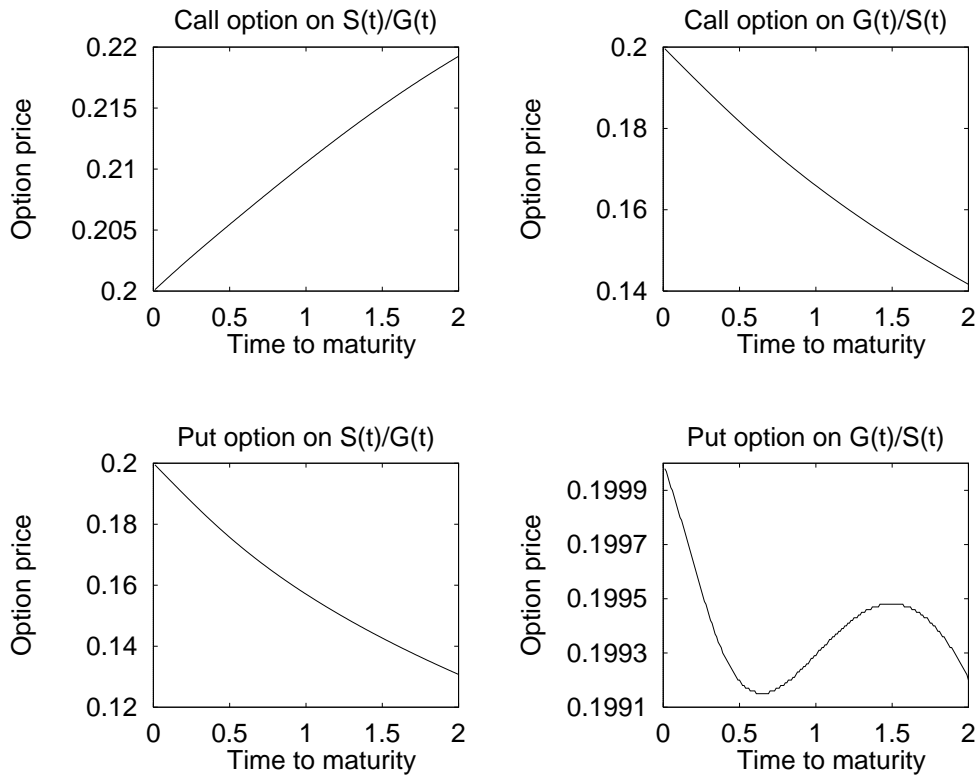


Figure 3: **Geometric Australian option prices as a function of time to maturity.** The exercise price is $K = 0.8$ for calls and $K = 1.2$ for puts. The other parameter values are: $r = 0.1$, $q = 0.03$, $\sigma = 0.2$, and $n = \infty$.

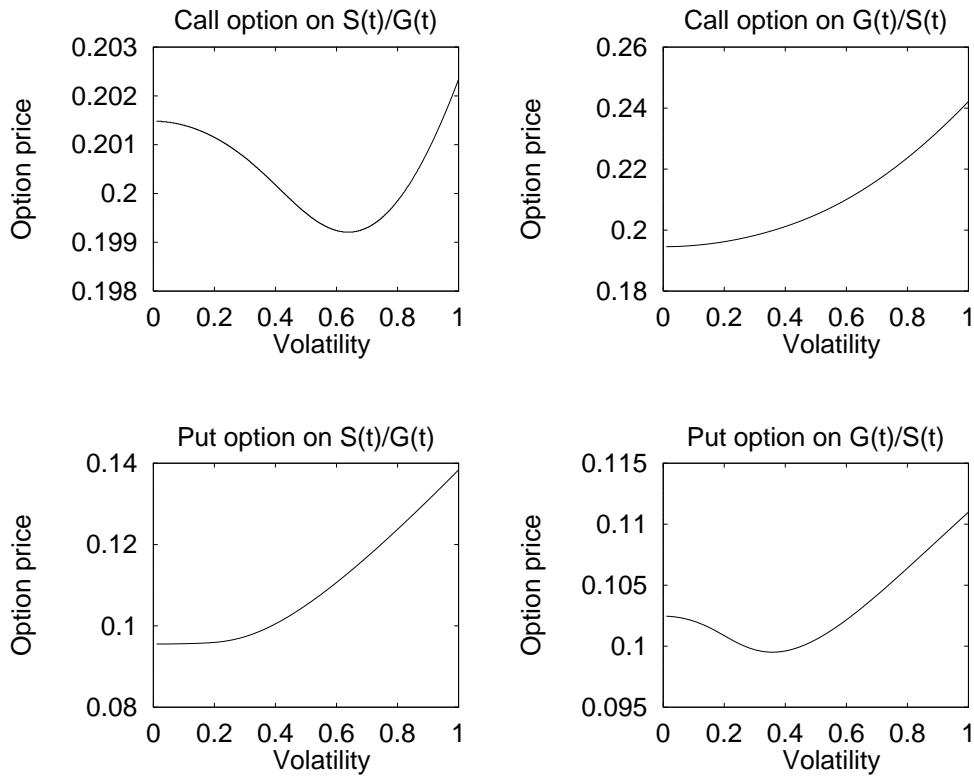


Figure 4: **Geometric Australian option prices as a function of volatility (σ).** The exercise price is $K = 0.8$ for calls and $K = 1.1$ for puts. The other parameter values are: $r = 0.1$, $q = 0.03$, $T = 0.1$, and $n = \infty$.

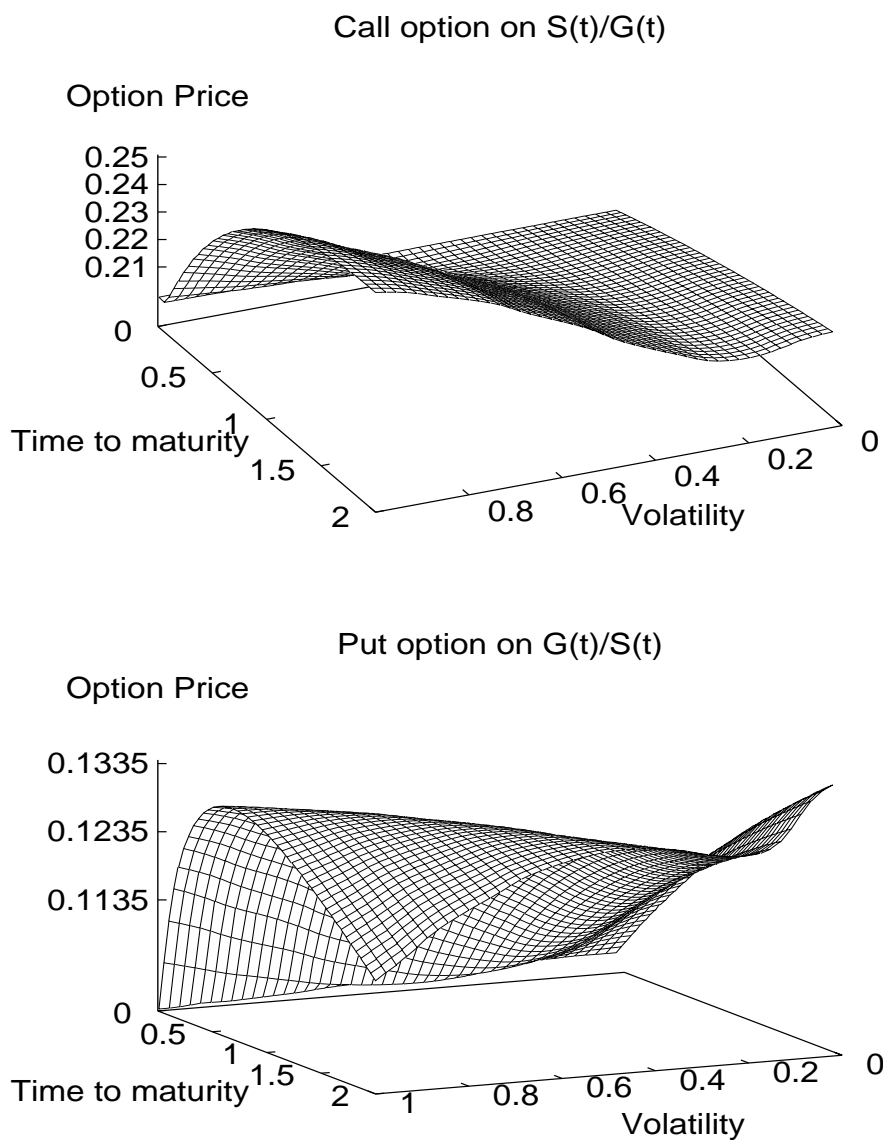


Figure 5: Geometric Australian option prices as a function of time to maturity and volatility (σ). The exercise price is $K = 0.8$ for the call option and $K = 1.1$ for the put option. The other parameter values are: $r = 0.1$, $q = 0.03$, and $n = \infty$.

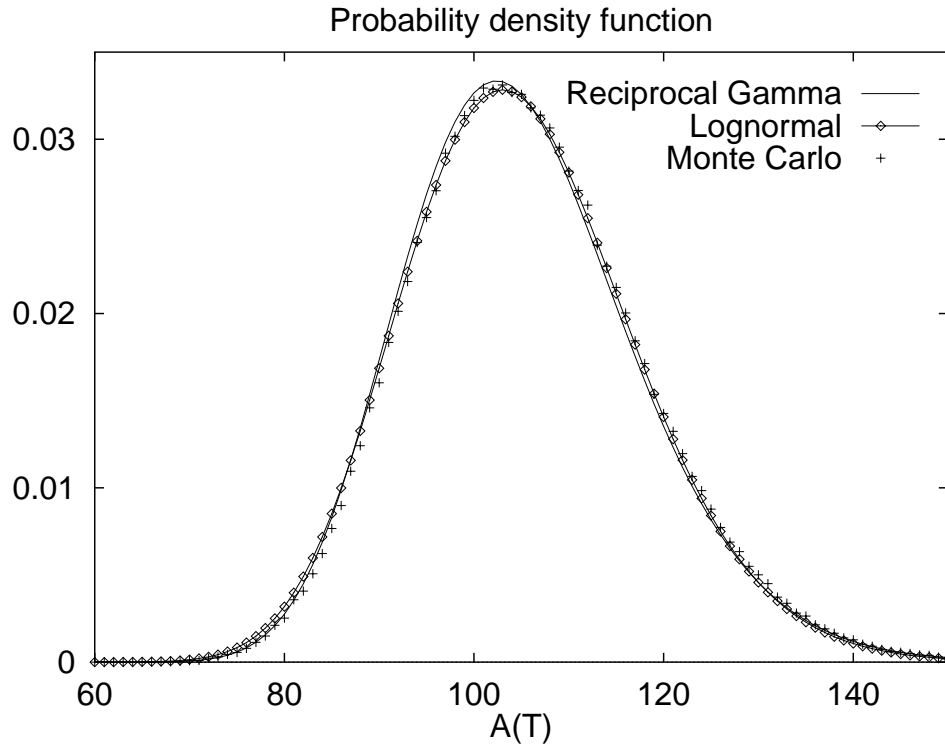


Figure 6: **Risk-neutral probability density function of the arithmetic stock price average at maturity.** The parameter values are: $r = 0.1$, $q = 0$, $\sigma = 0.2$, $T = 1$, $S_0 = 100$ and $n = \infty$. The expected value of the average price is 105.17, and the variance 152.74. For $n = \infty$ the true density function is reciprocal gamma, with parameters $\alpha = 74.42$ and $\beta = 1.29\text{E-}4$. This function is approximated with a lognormal distribution with the same moments. The density function is also estimated with Monte Carlo simulation, using a set of 50 runs of 10,000 paths with 1,000 time steps.