RANDOM INTERCEPT ITEM FACTOR ANALYSIS

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Abstract

The common factor model assumes a linear relation between the observed variables and a set of underlying latent traits. It also assumes that the linear coefficients, intercepts and slopes (factor loadings), linking the observed variables to the latent traits are fixed coefficients (i.e., common for all subjects). When the observed variables are subjects' direct responses to stimuli, such as their responses to the items of a questionnaire, the assumption of common linear coefficients may be too restrictive. This may occur, for instance if respondents to questionnaire items consistently use the response scale idiosyncratically. To account for this phenomenon we partially relax the fixed coefficients assumption by letting the intercepts in the factor model change across subjects while keeping the factor loadings fixed.

We show that, under suitable assumptions on this random component of the intercept, the covariance structure implied by a model with p factors and random intercepts is equivalent to the covariance structure model implied by a model with p + 1 factors model where one of the factors have common loadings.

Keywords

LISREL, random effects, difficulty factors, structural equation modeling, Life Orientation Test

1.Introduction

One of the most widely used statistical models in the Social Sciences is without doubt the common factor model. In this model a set of observed variables is expressed as a linear function of a smaller set of unobserved variables (latent traits) plus some error. Furthermore, the model specifies that the linear coefficients are fixed. Often times, this model is applied to the responses of a population of subjects to a set of stimuli. The typical example is when we wish to measure some broadly defined construct by means of a questionnaire composed of items for which some graded response scale is provided. Likert (1932) showed that the simple scoring system of assigning consecutive integers to the graded responses could not be outperformed by more sophisticated procedures. It has become standard practice to use this scoring system and to denote this type of items Likert-type items.

When the factor model is applied to Likert-type items, we are assuming that the relationship between the items and the underlying latent traits is linear. Also, because in this model the linear coefficients (intercepts and factor loadings) are fixed, we are in fact assuming that the expected relationship between a subject's latent trait and his/her responses to an item is the same for all respondents with a fixed level in those latent traits. The latter may be an unreasonable assumption as it is common to observe that some subjects consistently use only the extremes of the graded response scale, yet others use only the middle points of the scale, while others use the full range of the scale, and this idiosyncratic use of the response scale does not seem to be related to the subjects' level on the latent trait being measured. The purpose of this paper is to extend the common factor model to accommodate this phenomenon by adding a random component to the intercept thus treating the intercepts as a mixed effect rather than as a fixed effect.

The remaining of this paper is organized as follows. First, we shall review the common factor model with fixed coefficients. Next, we shall introduce our random intercept version of this model. We will show that by introducing suitable assumptions on this model, the random intercept factor model is equivalent to a factor model having an additional factor with common factor loadings which is orthogonal to all 'substantive' latent traits. Hence, this random intercept factor model can be fitted using a standard programs for covariance structure analysis. To illustrate our presentation we apply a random intercept factor model to the Life Orientation Test (LOT: Scheier & Carver, 1985).

2. The common factor model

Suppose a set of n Likert-type items has been administered to a random sample of N respondents. For ease of exposition and without loss of generality, we shall assume that each of the Likert-type items consists of the same number of categories, k. We shall also assume that successive integers $\{0, 1, 2, ..., k-1\}$ have been assigned to the subjects' responses. A p-dimensional factor model for these data is given by

$$y_{ij} = \gamma_{ij} + \boldsymbol{\lambda}_i'\boldsymbol{\eta}_j + e_{ij} \qquad i = 1, ..., n; \;\; j = 1, ..., N \tag{1}$$

where γ_{ij} denotes the intercept for item i and respondent j, $\boldsymbol{\lambda}_i$ denotes the vector of factor loadings for the same item, $\boldsymbol{\eta}_j$ denotes subject j's vector of factors, and e_{ij} denotes a residual. In the factor model it is further assumed that the intercept is common for all respondents

$$\gamma_{ij} = \mu_i \tag{2}$$

Thus, in this model the factors $\boldsymbol{\eta}$ and residuals $\mathbf{e}$ are random variables, whereas μ_i and $\boldsymbol{\lambda}_i$ are fixed constants common to all subjects.

It is convenient to express (1) and (2) in matrix form

$$\mathbf{y} = \boldsymbol{\gamma} + \boldsymbol{\Lambda}\boldsymbol{\eta} + \mathbf{e} \tag{3}$$

$$\boldsymbol{\gamma} = \boldsymbol{\mu} \tag{4}$$

Let the covariance matrix of the latent traits be denoted by $\boldsymbol{\Psi}$ and the covariance matrix of the residuals be denoted by $\boldsymbol{\Theta}$. The factor model further assumes that

1) The mean of the factors in the population of respondents is zero.
2) The mean of the residuals is zero.
3) The residuals are uncorrelated with each other, so that $\boldsymbol{\Theta}$ is a diagonal matrix.
4) The residuals are uncorrelated with the factors.

These assumptions, coupled with (3) and (4) imply the following well-known structure for the mean vector and covariance matrix of the observed variables

$$\boldsymbol{\Sigma}_y(\boldsymbol{\theta}) = \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta} \qquad \boldsymbol{\Sigma}_y(\boldsymbol{\theta}) = \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta} \tag{5}$$

where $\boldsymbol{\theta}$ denotes a parameter vector containing the distinct elements of $\boldsymbol{\Lambda}$, $\boldsymbol{\Psi}$, and $\boldsymbol{\Theta}$.

When no restrictions are imposed on $\boldsymbol{\mu}$ the most popular approach to estimate the parameter vector $\boldsymbol{\theta}$ is by minimizing the discrepancy function

$$F(\boldsymbol{\theta}) = \ln\left|\boldsymbol{\Sigma}_y(\boldsymbol{\theta})\right| - \ln|\mathbf{S}| + \mathrm{tr}\left(\mathbf{S}\boldsymbol{\Sigma}_y(\boldsymbol{\theta})^{-1}\right) - n \tag{6}$$

where $\mathbf{S}$ denotes the sample covariance matrix (see Browne and Arminger, 1995). When the observed data is multivariate normal, (6) yields maximum likelihood estimates of $\boldsymbol{\theta}$. However, when the observed data are responses to Likert-type items, it is unlikely that the distribution of the data is reasonably approximated by a multivariate normal distribution. If that is the case, then the normal theory standard errors and goodness of fit tests will be incorrect. With non-normal observations one can still obtain asymptotically correct standard errors and goodness of fit tests associated with (6) suitable to non-normal observations using results given by Arminger and Schoenberg (1989) and Satorra and Bentler (1994).

In any case, some restrictions on $\boldsymbol{\Lambda}$ and $\boldsymbol{\Psi}$ must be imposed to identify the model. Identification restrictions for restricted (a.k.a. confirmatory) models are given in Bollen (1989: pp. 238-251). When no prior knowledge is assumed, then an unrestricted factor model can be fitted (a.k.a. exploratory factor model) . The easiest way to identify an unrestricted model is to set $\boldsymbol{\Psi} = \mathbf{I}$ and letting $\boldsymbol{\Lambda}$ be a low echelon matrix (McDonald, 1999: p. 181), fixing to zero $\frac{p(p-1)}{2}$ factor loadings in the upper right corner of $\boldsymbol{\Lambda}$. The unrestricted solution may then be rotated to increase the interpretation of the unrestricted solution.

<u>The common factor model with a random intercept</u>

We shall now relax the assumption of an intercept common to all respondents by letting the intercept change from respondent to respondent. Instead of (2) we shall assume

$$\gamma_{ij} = \mu_i + \zeta_j . \tag{7}$$

That is now the intercept γ_{ij} consists of a fixed part μ_i that changes from item to item, and a random part ζ_j that changes from respondent to respondent. In matrix form we write (7) as

$$\boldsymbol{\gamma} = \boldsymbol{\mu} + \mathbf{1}\zeta \tag{8}$$

In this model, we let the variance of the random component of the intercept be φ and in addition to assumptions 1) to 4) above we assume that

5) The mean of the random component ζ of the intercept is zero in the population of respondents.
6) The random component ζ of the intercept is uncorrelated with the factors and with

the residuals.

Assumption 5) is imposed to identify the model, whereas assumption 6) reflects our observation that subjects' consistent patterns of usage of the response scale (for example, the use of the extreme values only, or of the middle values only) seem to be unrelated to the level of the respondents on the latent traits being measured.

In Appendix 1, we show that the model defined by (3) and (8), coupled with assumptions 1) through 6), implies the following structure for the mean vector and covariance matrix of the observed variables

$$\boldsymbol{\mu}_y = \boldsymbol{\mu} \qquad \boldsymbol{\Sigma}_y(\boldsymbol{\theta}) = \varphi \mathbf{1}\mathbf{1}' + \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta} \tag{9}$$

where $\boldsymbol{\theta}$ denotes a parameter vector containing φ and the distinct elements of $\boldsymbol{\Lambda}$, $\boldsymbol{\Psi}$, and $\boldsymbol{\Theta}$. This model is identified by the usual rules for the identification of the factor model. Furthermore, when no restrictions are imposed on $\boldsymbol{\mu}$, (9) can be estimated by minimizing a covariance structure discrepancy function such as (6) just like the common factor model.

To estimate (9) more easily using conventional software for covariance structure analysis, it is convenient to reparameterize the random intercept model as a common factor model

$$\mathbf{y} = \boldsymbol{\gamma} + \boldsymbol{\Lambda}^{*}\boldsymbol{\eta}^{*} + \mathbf{e} \qquad \boldsymbol{\gamma} = \boldsymbol{\mu} \tag{10}$$

where $\boldsymbol{\eta}^{*} = \left(\zeta \,\vdots\, \boldsymbol{\eta}\right)'$ and with

$$\boldsymbol{\Lambda}^{*} = \left(\sqrt{\varphi}\mathbf{1} \,\vdots\, \boldsymbol{\Lambda}\right) \qquad \mathrm{Cov}\left[\boldsymbol{\eta}^{*}\right] = \begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{\Psi} \end{pmatrix} \tag{11}$$

or alternatively, with

$$\boldsymbol{\Lambda}^{*} = \left(\mathbf{1} \,\vdots\, \boldsymbol{\Lambda}\right) \qquad \mathrm{Cov}\left[\boldsymbol{\eta}^{*}\right] = \begin{pmatrix} \varphi & 0 \\ 0 & \boldsymbol{\Psi} \end{pmatrix} \tag{12}$$

In Appendix 1 we show that the common factor model (10) with (11) or (12) also imply the mean and covariance structure (9). As a consequence, a common factor model with p latent traits and a random component in its intercept with Assumptions 5) and 6) can not be empirically distinguished from a common factor model with $p + 1$ latent traits whose first latent trait has a common factor loading and it is uncorrelated with all remaining p latent traits. In other words, if one finds the mean and covariance structure (9) to be a reasonable model for the observed data, the interpretation of the first latent trait as a random intercept or as a substantive latent trait must be made exclusively on substantive, not statistical, grounds. The random intercept factor model with assumptions 1) through 6) imply (9), but the converse is not true. This is of course true for any mean and covariance structure model (see Browne, 1982).

Most often, questionnaires consist of positively worded and negatively worded items where one of these two sets of items is inversely coded prior to analyzing the data. This is to avoid aquiescence effects on the respondents. The random intercept factor model is not invariant under such transformation and in Appendix 2 we show how to fit a random intercept factor model when a subset of the items has been inversely coded prior to the analysis.

3. An example: Modeling the LOT

The Life Orientation Test (LOT: Scheier & Carver, 1985), is a eight item questionnaire designed to measure optimism and pessimism. The response scale for the items is graded, consisting of five points. Four of the items are positively worded and are scored from 0 to 4, while the remaining four items are negatively worded and are coded from 4 to 0. The LOT was designed to measure a single dimension. However, several factor analytic studies (e.g., Scheier & Carver, 1985; Marshall, Wortman, Kusulas & Hervig, 1992; Chang, D'Zurilla & Maydeu-Olivares, 1994; Chang & McBride-Chang, 1996; Robinson, Kim, MacCallum & Kiecolt-Glaser, 1997) have revealed that a one factor model does not fit well this questionnaire. Instead, they found that a two factor model in which all positively worded items load on one dimension and all negatively worded items load on another dimension and both dimensions were correlated fitted well these data. However, we believe that it is hard to justify theoretically that optimism and pessimism are two distinct traits. Obviously, one may be optimistic about the outcome of a situation, and pessimistic about the outcome of another situation. But across situations (and the LOT measures generalized outcome expectancies) it is not clear how one can be both optimistic and pessimistic.

Here, we shall re-analyze Chang et al.'s (1994) data fitting the following factor models:

(A) a one dimensional model,

(B) an unrestricted two-dimensional model,

(C) a restricted two-dimensional model in which the positively worded items load on one dimension, the negatively worded items in another dimension and both dimensions are correlated, and

(D) a random intercept one factor model.

Note that (A) is a special case of (B), (C) and (D). Also, (C) and (D) are special cases of (B). Thus, can use tests for nested models to compare these pairs of models. We can not use a nested test to compare (C) and (C) as these models are not nested.

All models were estimated using LISREL 8.51 (Jöreskog & Sörbom, 2001) using the fitting function (6) with standard errors and a goodness of fit test asymptotically robust to non-normality. In Table 1, we give the item means, standard deviations and correlations. Positive and negatively worded items correlate negatively as the data was not recoded prior to analysis.

Insert Tables 1 to 2 about here

Goodness of fit tests are given in Table 2. The Satorra-Bentler (1994) scaled test statistics shown in this table suggest that the best model for this data is (C) a restricted two factor model. Nested tests (Satorra & Bentler, 2001) reveal that an unrestricted two-factor model does not fit better than this restricted model. No nested tests for the one factor model are presented as this model fits very poorly. The random intercept one factor model marginally fits this data and it is outperformed by an unrestricted two-factor model.

The fit of all these models is greatly improved when item 11 is removed from the LOT. Item 11 of the LOT, "I'm a believer in the idea that 'every cloud has a silver lining'", is a saying, the only saying in the inventory and this introduces a distortion in the subjects' responses. This can be seen in Table 3. Again, the one factor model fits very poorly, but now not only the unrestricted and restricted two-factor models fit these data well, but also does the one factor random intercept model. In fact, nested tests shown in this table reveal that the unrestricted two-factor model does not outperform neither the two-factor model nor the one factor random intercept model.

In Table 4 we provide the factor loadings for all the models considered when item 11 is removed from the LOT. We see in this table that the loadings for the one factor random intercept model are all large. Also, the estimate of the variance of the random component of the intercept is 0.13, rather small relative to the variance of the factor which was set to 1. However, the value of the variance of the random component of the intercept is rather large relative to its standard error, 0.01, and setting this variance equal to zero, which is equivalent to specifying a one factor model results in a very poorly fitting model.

In sum, we have shown that by introducing a random component in the threshold of a one factor model we do not reject Scheier and Carver's (1985) original hypothesis that optimism is indeed a one-dimensional construct. We find our solution more parsimonious than hypothesizing a two-dimensional factor model (e.g. Chang et al., 1994) or than fitting a one dimensional model with correlated errors (e.g., Scheier & Carver, 1985).

4. Discussion and conclusions

We have suggested a new covariance structure that applied researchers may wish to consider when fitting the factor model to questionnaire items. This covariance structure may be interpreted as arising from a factor model with a random component in its intercept. This model is obviously linked to the old literature on difficulty factors. A good review of this literature is given in McDonald and Ahlawat (1974; see also McDonald, 1999). This literature is concerned about needing additional dimensions using the common factor model that what could be expected from substantive theory. We have shown here that if indeed one is willing to assume that respondents to questionnaire items use the response scale idiosyncratically, and thus a random component needs to be incorporated into the intercept of the model, then an additional dimension will appear when fitting the data.

References

Arminger, G. & Schoenberg, R.J. (1989). Pseudo-maximum likelihood estimation and a test of mispecification in mean and covariance structure models. Psychometrika, 54, 409-425.

Bollen, K.A. (1989). Structural equations with latent variables. New York: Wiley.

Browne, M.W. (1982). Covariance structures. In D.M. Hawkins (Ed.). Topics in applied multivariate analysis (pp. 72-141). Cambridge: Cambridge University Press.

Browne, M.W. & Arminger, G. (1995). Specification and estimation of mean and covariance structure models. In G. Arminger, C.C. Clogg, and M.E. Sobel (Eds.) Handbook of statistical modeling for the social and behavioral sciences (pp. 185-249). New York: Plenum Press.

Chang, E.C., D'Zurilla, T.J., & Maydeu-Olivares, A. (1994). Assessing the dimensionality of optimism and pessimism using a multimeasure approach. Cognitive Therapy and Research, 18, 143-160.

Chang, L. & McBride-Chang, C. (1996). The factor structure of the Life Orientation Test. Educational and Psychological Measurement, 56, 325-329.

Jöreskog, K.G. & Sörbom, D. (2001). LISREL 8.51. Chicago, IL: Scientific Software.

Likert, R. (1932). A technique for the measurement of attitudes. Archives of Psychology, 140, 55.

Marshall, G.N., Wortman, C.B., Kusulas, J.W., Hervig, L.K., & Vickers, R.R. (1992). Distinguishing optimism from pessimism: Relations to fundamental dimensions of mood and personality. Journal of Personality and Social Psychology, 62, 1067-1074.

McDonald, R.P. (1999). Test theory. A unified treatment. Mahwah, NJ: Lawrence Erlbaum.

McDonald, R.P. & Ahlawat, K.S. (1974). Difficulty factors in binary data. British Journal of Mathematical and Statistical Psychology, 27, 82-99.

Robinson-Whelen, S., Kim, C, MacCallum, R.C., Kiecolt-Glaser, J.K. (1997). Distinguishing optimism from pessimism in older adults: Is it more important to be optimistic or not to be pessimistic? Journal of Personality and Social Psychology, 73, 1345-1353.

Satorra, A. & Bentler, P.M. (1994). Corrections to test statistics and standard errors in covariance structure analysis. In A. Von Eye and C.C. Clogg (Eds.). Latent variable analysis. Applications for developmental research (pp. 399-419). Thousand Oaks, CA: Sage.

Satorra, A. & Bentler, P.M. (2001). A scaled difference chi-square test statistic for moment structure analysis. Psychometrika, 66, 507-514.

Scheier, M.F., & Carver, C.S. (1985). Optimism, coping, and health: Assessment and implications of generalized outcome expectancies. Health Psychology, 4, 219-247.

Table 1

Means, standard deviations and correlations for the LOT data

Corr.	item 1	item 4	item 5	item 11	item 3	item 8	item 9	item 12
item 1	1.00							
item 4	0.51	1.00						
item 5	0.44	0.53	1.00					
item 11	0.25	0.34	0.22	1.00				
item 3	-0.16	-0.22	-0.26	-0.11	1.00			
item 8	-0.28	-0.38	-0.33	-0.19	0.50	1.00		
item 9	-0.24	-0.29	-0.30	-0.22	0.51	0.70	1.00	
item 12	-0.22	-0.35	-0.30	-0.26	0.44	0.54	0.52	1.00
Mean	2.24	2.40	2.56	2.34	1.85	1.39	1.32	1.40
Std	1.00	0.99	0.99	0.99	1.05	1.03	1.00	1.07

Notes: $N = 389$

Table 2

Goodness of fit tests for the LOT data

Absolute goodness of fit tests

Label	Model	$N\hat{F}$	$\hat{T}_S$	df	p
A	One factor model	217.40	211.46	20	<0.01
B	Unrestricted two-factor model	17.51	14.36	13	0.35
C	Restricted two-factor model	28.66	22.20	19	0.27
D	Random intercept one factor model	43.31	34.29	19	0.02

Nested goodness of fit tests

Comparison	$\hat{T}_{dif}$	df	p
C vs. B	7.93	5	0.16
D vs. B	19.36	5	<0.01

Notes: $N\hat{F}$ = minimum of the fitting function times sample size, $\hat{T}_S$ = Satorra-Bentler's (1994) scaled test statistic, $\hat{T}_{dif}$ = Satorra-Bentler's (2001) scaled test statistic for nested tests, p = p-value associated with Satorra-Bentler's statistic.

Table 3

Goodness of fit tests for the LOT data after removing item 11

Absolute goodness of fit tests

Label	Model	$N\hat{F}$	$\hat{T}_S$	df	p
A	One factor model	185.78	161.05	14	<0.01
B	Unrestricted two-factor model	8.31	6.40	8	0.60
C	Restricted two-factor model	16.99	12.63	13	0.48
D	Random intercept one factor model	19.63	15.52	13	0.13

Nested goodness of fit tests

Comparison	$\hat{T}_{dif}$	df	p
C vs. B	6.11	5	0.30
D vs. B	9.35	5	0.10

Notes: $N\hat{F}$ = minimum of the fitting function times sample size, $\hat{T}_S$ = Satorra-Bentler's (1994) scaled test statistic, $\hat{T}_{dif}$ = Satorra-Bentler's (2001) scaled test statistic for nested tests, p = p-value associated with Satorra-Bentler's statistic.

Table 4

Factor loadings for the LOT data after removing item 11

	one factor model	unrestricted two-factor model		restricted two-factor model		one factor random intercept
	$\boldsymbol{\lambda}$	$\boldsymbol{\lambda}_1$	$\boldsymbol{\lambda}_2$	$\boldsymbol{\lambda}_1$	$\boldsymbol{\lambda}_2$	$\boldsymbol{\lambda}$
item 1	0.38 (0.06)	0.65 (0.05)	0	0.64 (0.05)	0	0.54 (0.05)
item 4	0.48 (0.06)	0.80 (0.05)	-0.03 (0.08)	0.74 (0.05)	0	0.66 (0.05)
item 5	0.46 (0.06)	0.66 (0.06)	-0.08 (0.07)	0.65 (0.05)	0	0.61 (0.05)
item 3	-0.64 (0.06)	-0.28 (0.08)	0.59 (0.06)	0	0.65 (0.06)	-0.56 (0.06)
item 8	-0.86 (0.04)	-0.45 (0.08)	0.74 (0.06)	0	0.88 (0.05)	-0.79 (0.05)
item 9	-0.79 (0.05)	-0.35 (0.08)	0.76 (0.05)	0	0.82 (0.05)	-0.71 (0.05)
item 12	-0.70 (0.06)	-0.41 (0.08)	0.56 (0.07)	0	0.70 (0.06)	-0.65 (0.06)

Notes: $N = 389$; Standard errors for estimated parameters in parentheses. The correlation between the factors in the restricted two factor model was -0.53 (SE = 0.06). The variance of the random intercept was 0.13 (SE = 0.01).

Appendix 1

Mean and covariance structure implied by the random intercept factor model

Let $\boldsymbol{\Psi} = \mathrm{Cov}\left[\boldsymbol{\eta\eta}'\right]$, $\boldsymbol{\Theta} = \mathrm{Cov}\left[\mathbf{ee}'\right]$ and $\varphi = \mathrm{Cov}\left[\zeta\zeta'\right]$ (a scalar). We can re-write assumptions 1) to 6) as: 1) $\mathrm{E}\left[\boldsymbol{\eta}\right] = \mathbf{0}$, 2) $\mathrm{E}\left[\mathbf{e}\right] = \mathbf{0}$, 3) $\boldsymbol{\Theta}$ diagonal, 4) $\mathrm{Cov}\left[\boldsymbol{\eta}\mathbf{e}'\right] = \mathbf{0}$, 5) $\mathrm{E}\left[\zeta\right] = 0$, 6) $\mathrm{Cov}\left[\zeta\boldsymbol{\eta}'\right] = \mathbf{0}$ and $\mathrm{Cov}\left[\zeta\mathbf{e}'\right] = \mathbf{0}$.

Now, putting together (3) and (8) we have $\mathbf{y} = \boldsymbol{\mu} + \mathbf{1}\zeta + \boldsymbol{\Lambda\eta} + \mathbf{e}$. Then, the mean structure for $\mathbf{y}$ implied by the random intercept factor model is

$$\mathrm{E}\left[\mathbf{y}\right] = \boldsymbol{\mu} + \mathbf{1}\,\mathrm{E}\left[\zeta\right] + \boldsymbol{\Lambda}\,\mathrm{E}\left[\boldsymbol{\eta}\right] + \mathrm{E}\left[\mathbf{e}\right] = \boldsymbol{\mu}.$$

To obtain the covariance structure implied by this model, we note that by assumptions 1) through 6),

$$\mathrm{Cov}\left[\mathbf{yy}'\right] = \mathbf{1}\,\mathrm{Cov}\left[\zeta\zeta'\right]\mathbf{1}' + \boldsymbol{\Lambda}\,\mathrm{Cov}\left[\boldsymbol{\eta\eta}'\right]\boldsymbol{\Lambda}' + \mathrm{Cov}\left[\mathbf{ee}'\right] = \varphi\mathbf{11}' + \boldsymbol{\Lambda\Psi\Lambda}' + \boldsymbol{\Theta}$$

We shall now show that the factor model (10) with (11) or (12) implies the mean and covariance structure (9). We first note that by assumptions 1) through 4), $\mathrm{E}\left[\boldsymbol{\eta}^*\right] = \mathbf{0}, \mathrm{Cov}\left[\boldsymbol{\eta}^*\mathbf{e}'\right] = \mathbf{0}, \mathrm{E}\left[\mathbf{e}\right] = \mathbf{0}, \mathrm{Cov}\left[\mathbf{ee}'\right] = \boldsymbol{\Theta}$. Thus, $\mathrm{E}\left[\mathbf{y}\right] = \boldsymbol{\mu} + \boldsymbol{\Lambda}^*\,\mathrm{E}\left[\boldsymbol{\eta}^*\right] + \mathrm{E}\left[\mathbf{e}\right] = \boldsymbol{\mu}$. As for the covariance matrix implied by (10) we have $Cov\left[\mathbf{yy}'\right] = \boldsymbol{\Lambda}^* Cov\left[\boldsymbol{\eta}^*\boldsymbol{\eta}^{*\prime}\right]\boldsymbol{\Lambda}^{*\prime} + Cov\left[\mathbf{ee}'\right]$, which using (11) or (12) simplifies to the covariance structure given in (9).

Appendix 2

Effect of inverse coding a subset of items onto the random intercept factor model

Let $\mathbf{y}$ denote the observed variables prior to recoding, and $\mathbf{z}$ denote the recoded variables. We shall partition $\mathbf{z}$ into $\mathbf{z}_1$ and $\mathbf{z}_2$ where $\mathbf{z}_1$ denotes the set of items that are not recoded, $\mathbf{z}_1 = \mathbf{y}_1$, and $\mathbf{z}_2$ denote the set of items that are inversely coded, $\mathbf{z}_2 = (k-1)\mathbf{1} - \mathbf{y}_2$. The mean and covariance structure under the random intercept factor model for the original variables are $\boldsymbol{\mu}_y = \boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_y = \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}$, where we partition $\boldsymbol{\mu}$ and $\boldsymbol{\Lambda}$ according to the partitioning of $\mathbf{y}$ as $\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}$ and $\boldsymbol{\Lambda} = \begin{pmatrix} \mathbf{1} & \boldsymbol{\Lambda}_1 \\ \mathbf{1} & \boldsymbol{\Lambda}_2 \end{pmatrix}$, and $\boldsymbol{\Psi}$ is given by (12).

Now, the inverse coding transformation is

$$\begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ (k-1)\mathbf{1} \end{pmatrix} + \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}.$$

Thus mean and covariance structure under the random intercept factor model for the recoded variables are $\boldsymbol{\mu}_z = \boldsymbol{\mu}^*$ and $\boldsymbol{\Sigma}_z = \boldsymbol{\Lambda}^*\boldsymbol{\Psi}\boldsymbol{\Lambda}^{*\prime} + \boldsymbol{\Theta}$, where

$$\boldsymbol{\mu}^* = \begin{pmatrix} \boldsymbol{\mu}_1 \\ (k-1)\mathbf{1} - \boldsymbol{\mu}_2 \end{pmatrix} \qquad \boldsymbol{\Lambda}^* = \begin{pmatrix} \mathbf{1} & \boldsymbol{\Lambda}_1 \\ -\mathbf{1} & -\boldsymbol{\Lambda}_2 \end{pmatrix}.$$

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