# PORTFOLIO DELEGATION UNDER SHORT-SELLING CONSTRAINTS*

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**Abstract**
We study delegated portfolio management when the manager's ability to short-sell is restricted. Contrary to previous results, we show that under moral hazard, linear performance-adjusted contracts *do* provide portfolio managers with incentives to gather information. We find that the risk-averse manager's effort is an increasing function of her share in the portfolio's return. This result affects the risk-averse investor's choice of contracts. Unlike previous results, the purely risk-sharing contract is now shown to be suboptimal. Using numerical methods we show that under the optimal linear contract, the manager's share in the portfolio return is higher than what it is under a purely risk sharing contract. Additionally, this deviation is shown to be: (i) increasing in the manager's risk aversion and (ii) larger for tighter short-selling restrictions. As the constraint is relaxed the deviation converges to zero.

**Keywords**
Third best effort, Linear performance-adjusted contracts, Short-selling constraints.

**JEL Classification Numbers**
D81, D82, J33

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# 1 Introduction

Investors delegate portfolio decisions to managers because of their alleged skill in gathering superior information on movements in security prices. When the manager's research activity is not observed, the investor could face problems associated with moral hazard. Then, it could be in the investor's interest to provide the manager with incentives to gather better information. In studying the nature of such incentive contracts, past literature has assumed the manager's portfolio choice to be unbounded. Yet, we seldom observe environments where the manager's portfolio choice is totally "unrestricted." Practices like borrowing money, margin purchases, short-selling or investment in derivative securities are usually restricted. Our purpose is to study the effect of such constraints on incentive provision.

We assume that the manager's ability to short-sell is restricted *and* that investors have to cope with moral hazard. Our primary interest is in the impact of short selling restrictions on the power of incentives provided by linear *symmetric* contracts. We report three main results. First (Corollary 2), linear performance-adjusted contracts *do provide managers with incentives* for gathering better information. Second (Proposition 4), we show that the manager's share in the portfolio return is *different from that under the purely risk sharing contract,* (we shall refer to the purely risk sharing contract as the first best contract).[1] Third, using numerical methods, we show that the manager's share in the optimal portfolio is *higher than that under the first best* and decreases as we relax the leverage constraint. We also present some additional results. In a scenario without moral hazard, but with short selling restrictions: (i) under the optimal linear contract, the manager's share in the portfolio is equal to the one under the first best contract(Proposition 4); (ii) linear contracts dominate quadratic contracts (Proposition 6, in Appendix A). With moral hazard and short selling restrictions, numerical methods show that, quadratic contracts dominate linear contracts only for certain parameter values (Table 2 in Appendix A).

We take restrictions on short selling as given. Almazán *et al* (2001) report that 70% of mutual funds explicitly state (in Form N-SAR handed to the SEC) that short selling is not permitted. The authors , however, assert that these restrictions are more than regulatory prohibitions. Hence, endogenizing short selling constraints may be a valuable line for future research.

Our main focus is on the incentives provided by linear symmetric contracts. Such contracts need not be optimal in the domain of all contracts and quadratic contracts are known to perform better than linear contracts in certain environments. We com-

[1] A "first best" corresponds to the situation where there is no moral hazard, i.e. the manager's effort is observable and verifiable by a third party, and there is no restriction on short selling. The optimal linear contract in such a scenario is purely risk sharing in nature. Hence, we call the purely risk sharing contract the 'first best'. In doing so we slightly abuse terminology because, under symmetric information, the contract specifies the manager's level of effort in addition to the share of portfolio returns.

pare linear and quadratic contracts in Appendix A.[2] There are two reasons for focusing on linear contracts in the main text of the paper. First, from an institutional point of view, the Security Exchange Commission (SEC) restricts compensation contracts in the mutual fund industry to only linear symmetric contracts. Second, restricting our domain to symmetric linear contracts provides us with the very well known "no-incentive" benchmark. When no restrictions on short-selling exist, Stoughton (1993) and Admati and Pfleiderer (1997) have shown that linear (fulcrum) contracts *fail* to affect the manager's decision to gather better information. In other words, the manager's optimal effort choice is independent of the contract she receives from the investor. As a consequence, the only role for the linear contract is to split risk efficiently between the manager and the investor: *a higher risk aversion of the former relative to the latter would then imply no performance adjustment component in managers fees.*

In contrast to the "no-incentive" result, our first result asserts that under moral hazard and finite short-selling bounds, linear contracts do provide the manager with incentives to gather better information. Both assumptions are necessary for this result. With moral hazard but no short-selling bounds, the no-incentive result prevails. With short-selling constraints but no moral hazard, incentives for performance are not required. Hence, as we show in Proposition 4, the first best split is optimal.

The intuition behind our first result is as follows. With no short selling constraints the manager is able to undo the effects of incentives by appropriate modifications of the portfolio. Hence, we get the "no incentive" result. With finite short selling bounds, no matter how large they are, the manager anticipates that with positive probability she shall not be able to form the portfolio of her choice. This leads her to reduce effort in gathering better information. Under such circumstances, by increasing the incentive fee the investor expands the *manager's* portfolio set, thereby partially undoing the effects imposed by short-selling bounds. This in turn, provides her with incentives for spending more effort.

Given the investor's utility function, the cost of increasing effort through linear contracts may be too high. As a result, the investor may simply desire to share risk through the first best sharing rule and ignore effort inducement. Our second result rules out such behavior: the first best sharing rule is never optimal.

We are not able to derive closed form solutions for the optimal linear contract.[3] Using numerical methods, we show that the manager's share in the portfolio is higher than in the first best. Importantly, this share converges to the first best level as

[2] We thank an anonymous referee for persuading us to carry out this exercise. We also show that linear contracts outperform quadratic contracts under symmetric information and leverage constraints. Deriving the form of optimal contract in this context could be of interest. The challenge emanates from solving for fixed points when the domain of contracts is left unrestricted. Even if the domain were to consist of only polynomial contracts, one cannot necessarily ensure continuity in the manager's best response function when returns to stocks are normally distributed.

[3] The optimal program of the investor requires that we integrate over a Chi-square distribution of degree one. To our knowledge, such integration can only be be performed numerically.

the bounds on short selling get relaxed. Thus, the "no-incentive" result is a *special case*. This final result can be interpreted as follows. In the constrained scenario, the performance adjustment fee plays an additional role beyond risk sharing, namely effort inducement. When the short-selling bounds shrink (making the restriction tighter) the volatility of the portfolio decreases as well since fewer "extreme" portfolios are feasible. If the investor does not increase the performance adjustment fee the manager will be under-exposed to management risk. As a consequence, effort will also decrease. The risk sharing and the effort inducement arguments are aligned in the same direction: the optimal incentive fee increases above the first best value. This effect is enhanced by the manager's risk-aversion: given a certain level of short-selling, the (percentage) deviation from the first best share increases as the manager's risk-aversion augments.

The rest of the paper is organized as follows. Section 2 introduces the basics of the model. We distinguish four possible scenarios, depending on the restrictions on portfolio choice (*constrained/unconstrained*) and the observability of effort (*public-information/moral hazard*). The optimal linear unconstrained contract under public-information is termed the first best. The second best scenario is reserved for one where there are no constraints on short selling but where the manager's effort is not observable. The third best scenario pertains to the one where constraints on short selling are exogenously imposed and the manager's effort is unobservable. Section 3 studies linear contracts. Here we study linear contracts without restrictions on portfolio choice, both in the first best and second best scenarios. The same analysis is repeated for constrained portfolio problems in Section 4. Section 4.1 presents numerical results on the optimal linear contract under limited leverage, i.e. on the third best contract. Linear and quadratic contracts are compared in Appendix A. All proofs are provided in Appendix B.

## 2 The model

A typical fund will inform the customer that managers (who are involved in investment research) are responsible for choosing each fund's investments. Customers may also be informed about how the managers are compensated. Given the information, the customer decides how much to invest in the fund. In this paper we shall abstract from the decision problem of the consumer. Instead, assuming that the interests of the customer and the fund owner are the same, we shall focus on the determination of the manager's compensation scheme by the owner of the fund. Slightly abusing terminology, we call the owner of the firm the investor.

Let the manager and the investor have preferences represented by exponential utility functions. Throughout the paper we will use $a > 0$ ($b > 0$) to denote the manager (investor) as well as her (his) *absolute* risk aversion coefficient.

The manager's investment opportunity set consists of two assets: a risky asset with net return $\tilde{x}$ and a riskless bond. Assume that $\tilde{x}$ is distributed as a standard normal variable. The distribution of the risky asset return and the return on the bond are public information. As in Heinkel and Stoughton (1994), the bond is taken as the benchmark portfolio against which the returns on the manager's portfolio are measured. The investment horizon is one period. At the beginning of the period, the investor transfers one unit of wealth to the manager who also receives a compensation contract from the investor. This contract sets the management fee as a percentage of the wealth under management and consists of two components: a fixed *flat* fee, denoted by $F$, and a *performance adjustment* fee. The performance adjustment rate is calculated as a percentage $\alpha$ of the portfolio's excess return over the net return of the benchmark (which by assumption is the bond). Denote such a contract as $(\alpha, F)$. Normalize the net return of the bond to zero. If the manager refuses the contract the game ends and she receives her reservation value (normalized to $-1$). If she accepts the contract, she puts in some effort $e$ which results in a signal $y$. The signal $y$ is a realization of random variable $\tilde{y}$.[4] After observing $y$, the manager forms a portfolio $\{\theta(y), 1-\theta(y)\}$ where $\theta(y)$ and $1-\theta(y)$ respectively denote the proportions invested by the manager in the risky asset and the bond. Conditional on the contract $(\alpha, F)$ and $\theta(y)$, the wealth of the manager and the investor are random outcomes $\tilde{W}_a(y)$ and $\tilde{W}_b(y)$ with associated utilities $U_a\left(\tilde{W}_a\right)$ and $U_b\left(\tilde{W}_b\right)$.

The variable $\tilde{y}$ is partially correlated with the stock's return, $\tilde{y} = \tilde{x} + \tilde{\epsilon}$ with $\tilde{\epsilon}$ the noise term. The return on the risky asset and the noise term are assumed to be uncorrelated. Let $\tilde{\epsilon} \sim \mathcal{N}(0, \sigma^2)$, with $\sigma^2 < \infty$ such that higher $\sigma^2$ implies a less precise signal.

Recall that the manager observes the signal after putting in her privately observed effort. The amount of effort is assumed to affect the precision of the signal. More concretely we assume that $\sigma^2 = e^{-1}$. Therefore, the signal's precision, $P(e) = \frac{e}{1+e}$, is an increasing and concave function of effort. On the other hand, effort is costly for the manager. With constant absolute risk aversion $a$, let $V(a,e)/a$ be the monetary value of the manager's disutility of effort $e$.

After receiving the signal the manager updates her beliefs about the distribution of the risky asset, such that $\tilde{x} \mid y \sim \mathcal{N}\left(\frac{e}{1+e}\, y, \frac{1}{1+e}\right)$.[5] Given these updated beliefs, the manager chooses $\theta(y)$. For any $(\alpha, F)$ and $\theta$, the conditional (net) wealth of the manager and the investor can be written as, respectively:

$$\tilde{W}_a(y) = F + \alpha\,\theta\,\tilde{x}\,|\,y, \tag{1}$$
$$\tilde{W}_b(y) = (1-\alpha)\,\theta\,\tilde{x}\,|\,y - F. \tag{2}$$

The utilities of the investor and the portfolio manager are given by, respectively,

[4] We will follow the standard notation whereby a symbol with a tilde on top will represent the variable and the same symbol, without a tilde, its realization.

[5] The vertical bar reads as "conditional to."

$U_a\left(\tilde{W}_a\right) = -\exp(-a\tilde{W}_a + V(a,e))$ and $U_b\left(\tilde{W}_b\right) = -\exp(-b\tilde{W}_b)$. We assume the function $V(a,e)$ is continuous and twice differentiable, with continuous derivatives. Moreover, the function is assumed to satisfy:[6]

**Assumption (S1)** $V(a,0) = V'(a,0) = 0$

**Assumption (S2)** $V'(a,e) > 0$ for all $e > 0$

**Assumption (S3)** $\frac{V''(a,e)e}{V'(a,e)} > P(e)$ for all $e > 0$

Assumptions (S1) and (S2) are standard in the literature. Assumption (S3) sets an upper bound to the signal's precision: the marginal cost of effort must increase *fast enough*. This will guarantee the existence of an optimal effort level for the manager. This assumption discards, for instance, linear disutility functions. Any quadratic function of effort that satisfies (S1) and (S2) will verify (S3) as well.

# 3 Unconstrained linear contracts

Assume that the manager's effort decision is publicly observable and let the investor's choice be restricted to linear contracts. Given the negative exponential utility functions for both the investor and the manager, the Pareto efficient sharing rules are linear -see Wilson (1968). Hence, each individual receives the fraction of the risky asset equal to his risk tolerance divided by the aggregate social risk tolerance. We will denote this result *the first best outcome*, $\alpha_{FB} = \frac{1}{1+r}$, where $r = \frac{a}{b}$ represents the manager's "relative" (to the investor) risk aversion.

To derive the non-incentive result, assume that the signal is observed only by the manager who decides, privately, how much effort to put in. Proceeding by backward induction, we first solve the manager's optimal portfolio problem. When the manager is *unconstrained* in her portfolio choice, she can select *any* $\theta$ from the real line $\Re$. Given some effort choice $e$ and some signal realization $y$, the manager chooses $\theta(y)$ to maximize her conditional expected utility of wealth $E\left[U_a\left(\tilde{W}_a(y)\right)\right]$ subject to $\theta(y) \in \Re$. Solving this we get:[7]

$$\theta(y) = \frac{e}{a\alpha}y. \tag{3}$$

Having solved for the manager's optimal portfolio problem, we now need to solve for her effort (previous stage) decision. Given (3), the manager forms her indirect unconditional utility function by taking expectations over $y$. This is written as $E\left[U_a\left(\tilde{W}_a(e)\right)\right] = -\exp\left(-aF + V(a,e)\right) \times g(e)$, where

[6] Prime ($\prime$) and double prime ($\prime\prime$) denote, respectively, first and second derivative with respect to effort.

[7] See Stoughton (1993).

$$g(e) = \left(\frac{1}{1+e}\right)^{1/2}. \tag{4}$$

Notice that *the manager's expected utility is independent of* $\alpha$. The expected utility maximizing effort solves the first order condition:

$$V'(a, e_{SB}) = \frac{1}{2(1+e_{SB})}. \tag{5}$$

Assumptions (S1)-(S3) guarantee the existence of $e_{SB} > 0$ satisfying equation (5). Note that the optimal effort in the second best scenario (which we call the second best effort) is a function only of the manager's risk aversion coefficient; in particular, *it does not depend on* $\alpha$ *or* $F$. This, in essence, is the non-incentive result.

Finally, in the first stage, the investor offers the manager a contract $(\alpha, F)$ that maximizes her expected utility subject to the manager's incentive compatibility constraint, (5), and the manager's participation constraint. Since $e_{SB}$ is unique with respect to $(\alpha, F)$, we can write the investor's utility as a function of $e_{SB}$ and solve for $(\alpha_{SB}, F_{SB}) \in \arg\max_{\alpha,F} E\left[U_b\left(\tilde{W}_b(\alpha, F, e_{SB})\right)\right]$ subject to the participation constraint $E\left[U_a\left(\tilde{W}_a(e_{SB})\right)\right] \geq -1$.

We define the functions $m(\alpha) \equiv \frac{1-\alpha}{r\alpha}$ and $M(\alpha) \equiv m(\alpha)\,(2 - m(\alpha))$. These functions will also help in later analysis. Let us denote $\Phi(x) = \int_0^x \phi(s)\,ds$, with $\phi(s) = \frac{1}{\sqrt{2\pi}}\, s^{-1/2}\exp(-s/2)$ when $s > 0$; $s = 0$ otherwise. $\Phi(\cdot)$ is the cumulative probability function of a Chi-square variable with one degree of freedom and $\phi(\cdot)$ is the corresponding density function.

With these definitions, Appendix B shows that the investor's expected utility can be written as

$$E\left[U_b\left(\tilde{W}_b(\alpha, F, e)\right)\right] = -\exp(aF/r)\left(\frac{1}{1+eM(\alpha)}\right)^{1/2}. \tag{6}$$

Since the manager's expected utility (4) is independent of $\alpha$, the optimal contract satisfies the first order condition $\frac{\partial}{\partial\alpha}M(\alpha_{SB}) = 0$. The function $M(\alpha)$ is concave for all $\alpha < 3/2(1+r)$, convex otherwise. Thus, given the later equation, it follows that $\alpha_{SB} = \frac{1}{1+r}$ is the (unique) solution to the investor's problem. The reader can verify that this result *corresponds to the first best share of risk.* In the second best, unrestricted scenario, the first best split prevails in spite of the asymmetry in information. Finally note that when $b$ tends to zero $\alpha_{SB}$ tends to zero and hence the *performance adjustment fee (captured by* $\alpha$*) has no role.*

Replacing $\alpha_{SB} = \alpha_{FB}$ in (6) and provided that the manager's participation constraint is binding in the optimum, the investor's expected utility in the unconstrained linear scenario will be given by:

$$E\left[U_b\left(\tilde{W}_b(e)\right)\right] = -\exp(V(a,e)/r)\left(\frac{1}{1+e}\right)^{\frac{a+b}{2a}}. \tag{7}$$

Maximizing the latter expression with respect to effort, we obtain the first best effort condition:

$$V'(a, e_{FB}) = (1+r)\frac{1}{2(1+e_{FB})}. \tag{8}$$

Comparing (5) with condition (8), it follows that the second best effort is always smaller than the first best effort.

# 4 Constrained linear contracts

We now study the effort and portfolio decisions of a manager who, unlike in the previous section, is *restricted in her portfolio choice*. We will distinguish between a *constrained public-information scenario* (where the manager's effort decision is publicly observable) and a *third best scenario*, where the manager's effort decision is private. In this scenario we will also analyze the effect of the restriction on the investor's optimal linear contract problem.

The restriction, that we call "bounded short-selling" [BSS], can be expressed as $|\theta| \leq \kappa$, $1 \leq \kappa < \infty$. The symmetry with respect to $\kappa$ is convenient in order to simplify the algebra.[8] Note that $\kappa$ can be any large number. All we require is that it should not be infinite.

Recall that $\theta$ and $1-\theta$ denote, respectively, the proportions invested by the manager in the risky asset and the bond. Also, in our model, the bond is taken as the benchmark portfolio. So, given the contract $(\alpha, F)$, the [BSS] restriction can be interpreted as a constraint on the manager's "personal" portfolio, $\{\alpha\theta, \alpha(1-\theta)\}$, as well as a constraint on the portfolio leverage. For instance, if $\kappa = 1$, [BSS] implies that the maximum short-selling allowed is 100% of the initial wealth ($\theta \geq -1$). Symmetrically, it also implies that $1-\theta \leq 2$. Hence, the maximum amount of money the manager is allowed to hold in the benchmark is $2\alpha$ (in our model, the initial wealth is normalized to 1 unit).

We start by providing an intuitive answer to the following question: How does our restriction influence the manager's effort decision? Increasing effort expenditure

[8]Note that $\tilde{y}$ has a *normal* distribution. None of our results depends, qualitatively, on this assumption.

implies that the signal's precision becomes sharper. However, introducing [BSS] "distorts" the manager's portfolio decision: for certain signals, the manager may not be able to form the portfolio of her choice. From an ex-ante perspective, the net effect of this trade-off results in a decrease in the marginal utility of effort as compared to the case where [BSS] does not hold. As a consequence, $\alpha$ now plays an additional role: by increasing $\alpha$ the investor can "marginally" relax the restriction imposed by [BSS]. Hence, a higher $\alpha$ induces the manager to exert higher effort.

Based on the above intuition, it follows that the manager's optimal effort under [BSS] will be: (i) smaller than $e_{SB}$ for all $\alpha$ and (ii) increasing in $\alpha$. Also, the distortion between the two effort levels should be inversely related to the manager's risk aversion: i.e. the larger is $a$ the smaller is the effect of [BSS] on the manager's effort decision. In the limit, when either $\kappa$ or $a$ tend to infinity, the effect of the restriction should vanish and we should return to the second best. In what follows, we formalize this intuition.

As in Section 2, we proceed by backward induction. The manager's optimal portfolio solves the following "constrained" problem $\theta(y) = \arg\max_\theta E\left[U_a\left(\tilde{W}_a(y)\right)\right]$ subject to $\kappa \geq \theta \geq -\kappa$. Let $\lambda_l \geq 0$ (lower bound) and $\lambda_u \geq 0$ (upper bound) denote the corresponding Lagrangian multipliers, such that, at the optimal $(\theta+\kappa)\lambda_l = 0$ and $(\theta-\kappa)\lambda_u = 0$.

Conditional on the signal realization $y$, and a given level of effort $e$, there are three possible solutions: (i) If $\lambda_u = 0$ and $\lambda_l = -\frac{e}{1+e}\alpha\left(y + \frac{\kappa a\alpha}{e}\right) > 0$, then short-selling is at the maximum and $\theta(y) = -\kappa$; (ii) if $\lambda_l = 0$ and $\lambda_u = \frac{e}{1+e}\alpha\left(y - \frac{\kappa a\alpha}{e}\right) > 0$, then leverage is at the maximum and $\theta(y) = \kappa$. Otherwise, $\lambda_l = \lambda_u = 0$, and the optimal portfolio is $\theta(y) = \frac{e}{a\alpha}y$.

The latter, "interior" solution coincides with the manager's optimal portfolio (3) in the unconstrained problem. The dollar amount, $\alpha\theta$, invested in the risky asset by the manager in her "personal" portfolio is independent of $\alpha$. In the "corner" solutions the dollar amount invested ($\alpha\kappa$) or sold short ($-\alpha\kappa$) in the risky asset is, in absolute value, increasing in $\alpha$: the manager will "behave" indeed as an investor with decreasing absolute risk aversion.

Writing the optimal portfolio as a function of the signal $y$, we have:

$$\theta(y) = \begin{cases} -\kappa & \text{if } y < -\frac{\kappa a\alpha}{e} \\ \frac{e}{a\alpha}y & \text{if } |y| \leq \frac{\kappa a\alpha}{e} \\ \kappa & \text{if } y > \frac{\kappa a\alpha}{e}. \end{cases} \tag{9}$$

We are now in a position to solve for the manager' choice of effort. Let us first investigate the manager's utility of effort. Recall that the manager had accepted some contract $(\alpha, F)$ in the beginning of the game. To decide on how much effort to put in she uses the knowledge that for each $y$ that she observes in the future, she will form the portfolio $\theta(y)$. Replacing the optimal portfolio $\theta(y)$ in the manager's conditional expected utility function and taking expectations over $y$ we arrive at the manager's unconditional expected utility function.

**Proposition 1** *Given the contract* $(\alpha, F)$ *and the constraint* $\kappa < \infty$, *the expected utility function of the risk-averse manager is*

$$E\left[U_a\left(\tilde{W}_a(\alpha, F, e\,|\,\kappa)\right)\right] = -exp(-aF + V(a,e)) \times g_\kappa(e\,|\,\alpha), \text{ with } g_\kappa(e\,|\,\alpha) =$$

$$\left(\frac{1}{1+e}\right)^{1/2} \times \Phi\left(\frac{(\kappa a\alpha)^2}{e}\right) + exp\left(\frac{(\kappa a\alpha)^2}{2}\right) \times \left(1 - \Phi\left(\frac{(\kappa a\alpha)^2}{e}(1+e)\right)\right) \tag{10}$$

*a decreasing and convex function of effort e.*

Equation (10) confirms the intuition presented at the beginning of this section. The unconditional expected utility of the constrained manager (i.e. after introducing [BSS]) can be expressed as the weighted sum of two utility functions.[9] The first function corresponds to the "interior" expected utility in (4) where the manager is not affected by the constraint. The second function is the manager's expected utility when the constraint is binding. In that case the manager sets $|\theta| = \kappa$. Note that, unlike the unconstrained case, $g_\kappa(e\,|\,\alpha)$ *depends on* $\alpha$. So, an interesting question is: how will changes in $\alpha$ affect the manager's utility? Corollary 1 answers this question.

**Corollary 1** *Given some contract* $(\alpha, F)$ *and the constraint* $\kappa < \infty$, *the manager's unconditional expected utility is increasing in* $\alpha$. *In the limit, when either the constraint,* $\kappa$, *or the manager's risk aversion coefficient,* $a$, *tend to infinity, the marginal utility of* $\alpha$ *is zero.*

Note that the second part of the corollary derives the *no-incentive* result as a special case of our model. To see the intuition behind the corollary, let us rewrite the constraint [BSS], given (3), as follows:

$$|y|\, e \leq \kappa a\alpha. \tag{11}$$

The left-hand term represents the risky asset's conditional mean return (absolute value) weighted by its precision. The right-hand side term is the short-selling limit, $\kappa$, multiplied by the manager's risk aversion coefficient weighed by $\alpha$. Clearly, as long as $|y| < \kappa a\alpha/e$, the manager's optimal decision will not be affected by [BSS]. In this

[9] The disutility function, $V(a,e)$, affects both terms. This is because the effort decision is taken *ex-ante*, before the signal is observed. Note that the weights are not constant: they are a function of effort themselves.

case, the marginal utility of $\alpha$ is zero and the manager's effort decision is independent of the contract. However, when the signal exceeds either bound (i.e. for "very good" or "very bad" signals) the manager would want to invest in her portfolio more than she is allowed to. Clearly, such a distorting effect will diminish as $\alpha$ and/or the risk aversion $a$ increase. So, for all $a < \infty$, the manager's marginal utility of $\alpha$ is positive. In the limit, when the right-hand side term in (11) tends to infinity the restriction vanishes and (10) converges towards the unconstrained utility function (4).

We now consider the manager's choice of effort. The manager chooses effort to maximizes her unconditional expected utility. Given $(\alpha, F)$, the manager's (third best) effort solves:

$$e_{TB}(\alpha) = \arg\max_{e \geq 0} -\exp(-aF + V(a, e)) \times g_\kappa(e \,|\, \alpha). \tag{12}$$

We are interested in the properties of the third best effort. Note that, unlike in the unconstrained second best case, *effort now depends on $\alpha$*. Corollary 1 had shown that the utility of the constrained manager increases in $\alpha$: by increasing the performance adjustment fee in the contract, the investor allows the manager to get "marginally" closer to her optimal unconstrained personal portfolio. The investor can now exploit this phenomenon to influence the manager's effort choice. In fact, effort turns out to be an increasing function of $\alpha$.

The intuition works as follows. Recall that the manager decides how much effort to exert after accepting the contract $(\alpha, F)$ and before receiving the signal. When the manager is unconstrained then, for any signal $y$, the absolute value of the manager's unconstrained portfolio (3) is increasing in effort. This marginal benefit is traded off against the inherent marginal disutility of effort to get at the second best level of effort. However, when the manager is constrained, equation (11) tells us that by exerting more effort the manager could actually "enhance" the distortion induced by [BSS]. Therefore, the marginal utility of effort and (hence) effort is lower than in the second best case.

**Proposition 2** *Given assumptions (S1)-(S3), the contract $(\alpha, F)$ and the constraint $\kappa < \infty$, there exists a unique $e_{TB}(\alpha) \geq 0$ that maximizes the manager's expected utility. Moreover, $e_{SB} > e_{TB}(\alpha)$ for all $\alpha \in [0, 1]$. Both are equal, in the limit, when either the constraint, $\kappa$, or the manager's risk aversion coefficient, $a$, tend to infinity.*

Now, following up with the argument in (11), a contract with a higher $\alpha$ marginally enlarges the manager's personal portfolio opportunity set: certain portfolios that were not feasible before turn now to be feasible. As a consequence, the marginal utility of effort increases. Thus, the optimal effort put by the manager is higher. In other words, the third best effort moves towards the second best.

**Corollary 2** *The manager's effort $e_{TB}(\alpha)$ is a continuous and differentiable function. Moreover, it is increasing in $\alpha$.*

We now turn now to the investor's (first stage) problem. First, we introduce the investor's unconditional utility function when the manager faces [BSS]. The constrained manager solves the restricted problem in Section 4 and her optimal portfolio is (9). Given (2), the investor's conditional utility function $E\left[U_b\left(\tilde{W}_b(y\,|\,\kappa)\right)\right]$ can be written as a function of $m(\alpha)$ and $M(\alpha)$ defined in Section 3. Following the same procedure we used to derive the manager's unconditional expected utility function, we arrive at the investors's expected utility function. It is stated in the following proposition.

**Proposition 3** *Under [BSS], for a given contract* $(\alpha, F)$, *the expected utility function of the risk-averse investor is* $E\left[U_b\left(\tilde{W}_b(\alpha, F, e\,|\,\kappa)\right)\right] = -exp(aF/r) \times f_\kappa(\alpha, e)$, *with*

$$f_\kappa(\alpha, e) = \left(\frac{1}{1+eM(\alpha)}\right)^{1/2} \times \Phi\left(\frac{(\kappa a\alpha)^2}{e}\frac{1+e\,M(\alpha)}{1+e}\right) + exp\left(\frac{(\kappa a\alpha\, m(\alpha))^2}{2}\right) \times \left(1 - \Phi\left(\frac{(\kappa a\alpha)^2}{e}\frac{(1+e\,m(\alpha))^2}{1+e}\right)\right). \tag{13}$$

After deriving the close-from solution to the investor's expected utility, we want to investigate how the presence of portfolio constraints *and* moral hazard affects the optimal linear contract. Assume first that the manager's effort decision were observable. In this case the investor maximizes his expected utility with respect to $\alpha$ and effort subject to the participation constraint $-\exp(-aF + V(a, e)) \times g_\kappa(e\,|\,\alpha) \geq -1$. Clearly, effort is not a function of $F$. This, along with the facts that the left-hand side is increasing in $F$ and the investor's utility is decreasing in $F$, implies that under the optimal contract the participation constraint is binding. So, the investor's problem is reduced to finding the optimal split and effort that maximize

$$E\left[U_b\left(\tilde{W}_b(\alpha, e\,|\,\kappa)\right)\right] = -\exp(V(a, e)/r) \times g_\kappa(e\,|\,\alpha)^{1/r} \times f_\kappa(\alpha, e). \tag{14}$$

On the other hand, when the manager's effort decision is not observable by the investor, the *third best problem* consists in finding the optimal split $\alpha_{TB}$ that maximizes (14) subject to the manager's optimal effort condition (12). Note that, due to first order condition (B7) in the Appendix B, (12) is uniquely solvable in terms of $\alpha$.

Despite this simplification, it is difficult to find a closed form solution for the optimal linear contract. Yet, we can still show that under [BSS] and in the absence of moral hazard, the first best risk-share is still optimal, consistently with the result in Haugen and Taylor (1987). On the contrary, in the presence of moral hazard, the optimal $\alpha_{TB}$ is no longer equal to $\alpha_{FB}$. This is to be expected because under [BSS] $\alpha$ plays an additional role over risk-sharing. As in most moral hazard problems, efficiency in risk allocation has to be traded off against effort inducement. These results are summarized in the following proposition.

**Proposition 4** *When the effort decision is public information, the first best risk share, $\alpha_{FB}$, is optimal under [BSS]. Moreover, for any finite $\kappa$, the investor's optimal effort choice is smaller than the first best effort. When $\kappa \to \infty$ both levels of effort coincide.*

*When the effort decision is not observable by the investor, the first best risk share, $\alpha_{FB}$, is not optimal under [BSS].*

## 4.1 A numerical solution to the linear third best contract

As mentioned in the previous section, it is difficult to solve analytically for the optimal contract. In this section we present a numerical solution for the third best contract. Our interest will pertain to the optimal third best share, $\alpha_{TB}$. We assume a quadratic disutility function of effort, $V(a, e) = a\,e^2$. Exercises will be carried out by setting the investor's *risk-tolerance* coefficient $(1/b)$ to 24. We will consider four different values for the manager's risk-tolerance coefficient $1/a = \{3, 8, 15, 24\}$. We will vary the short-selling/leverage constraint, $\kappa$, through 10 integer values, from 1 (tightest restriction, no leverage) through 10 (weakest restriction).

Given the disutility function, condition (5) implies that the second best effort of a manager with risk-tolerance coefficient $1/a$ is $e_{SB}(1/a) = \frac{1}{2}\left(\sqrt{1 + 1/a} - 1\right)$. Thus, for the four different values of the risk tolerance coefficient under study we obtain the corresponding values of $e_{SB}(1/a) = \{1/2, 1, 3/2, 2\}$. Note that the second best effort increases with the manager's risk tolerance.[10]

For each $\kappa$, the algorithm creates a grid of 99 values of $\alpha$ from 0.01 through 0.99. Condition (B7) in the Appendix B is solved for each pair $(\alpha, \kappa)$. That gives a numerical value of $e_{TB}$ for each pair $(\alpha, \kappa)$. The resulting matrices of third best efforts (which we do not report) confirm the predictions of Proposition 2 and Corollary 2: for all risk-aversion coefficients and all leverage bounds, the third best effort is (i) smaller than the corresponding second best effort and (ii) increasing in $\alpha$.

For each $\kappa$, the investor's expected utility (14) is evaluated across $\alpha$. Note that $e_{TB}$ and $F_{TB}$ as functions of $\alpha$ are implicitly taken into account in these calculations (the latter is a function of $\alpha$ due to the fact that the participation constraint is binding). Figure 1 plots the investor's expected utility function as a function of $\alpha$ for four values of $\kappa$ when $1/a = 1/b = 8$. In all cases, *the investor's expected utility as a function of $\alpha$ is concave*. In such a case, the proof of Proposition 4 implies that $\alpha_{TB} > \alpha_{FB}$.

The first row within each panel in Table 1 reports the values of $\alpha_{TB}(1/a, 1/b)$ which maximize the investor's expected utility for $1/b = 24$, $1/a = \{3, 8, 15, 24\}$ and $\kappa = 1, ..., 10$. In all cases, the figures illustrate an important numerical result:

[10] The region of "acceptable" *relative* risk aversion coefficients varies from source to source -see Mehra and Prescott (1985). Our manager's *expected relative risk aversion coefficient* is defined as her absolute risk aversion coefficient $a$ times the manager's unconditional expected portfolio wealth, $E_y(\alpha\theta\tilde{x}(y)) = \frac{e}{a}$. Thus, the values of $a$ are chosen so as to yield $e_{SB}(1/a) \in [1/2, 2]$.

$\alpha_{TB} > \alpha_{FB}$ in the constrained scenario. This, as mentioned earlier is a consequence of the concavity of the investor's utility function. Interestingly, as $\kappa$ increases (i.e., the constraint is relaxed) $\alpha_{TB}$ monotonically converges to $\alpha_{FB}$.

The relationship between the manager's risk-aversion and $\Delta\alpha/\alpha = \frac{\alpha_{TB}-\alpha_{FB}}{\alpha_{FB}}$, for different $\kappa$s, is reported in the second row of each panel in Table 1. We see that, for each $\kappa$, the difference in percentage is higher for higher values of the manager's risk-aversion. The difference can be very dramatic: it ranges from over 280% for $(1/a = 3, \kappa = 1)$ to 20% for $(1/a = 24, \kappa = 10)$.

These results suggest that benchmarked contracts may play a significant role in providing incentives to managers for exerting effort. When the short-selling bounds decrease (making the restriction tighter) the volatility of the portfolio decreases as well since fewer extreme portfolios are feasible. If the investor does not increase the performance adjustment fee the manager will be under-exposed to active management risk. As a consequence, effort will also decrease. The risk sharing and the effort inducement arguments are aligned in the same direction: the optimal performance adjustment fee increases. The change in $\alpha$ due to the incentive role is more visible the smaller the manager's risk tolerance because $\alpha_{FB}$ in that case is relatively smaller.

The third and fourth rows of each panel in Table 1 report the percentage difference in effort, $\Delta e/e = \frac{e_{TB}(\alpha_{TB})-e_{TB}(\alpha_{FB})}{e_{TB}(\alpha_{FB})}$, and certainty equivalent wealth, $\Delta C/C = \frac{C_{TB}(\alpha_{TB})-C_{TB}(\alpha_{FB})}{C_{TB}(\alpha_{FB})}$, in the constrained scenario.[11] Hence, the ratio $\Delta C/C$, can be interpreted as the net return (on the end-of-period wealth $C_{TB}(\alpha_{FB})$) that would compensate the investor for the lower utility of the suboptimal split, $\alpha_{FB}$, in the third best scenario.

The last column in Table 1 represents very relaxed constraints ($\kappa = 10$). Even here, $\Delta e/e$ is around 30% for the most risk averse manager. In all cases $\Delta e/e$ decreases with the manager's risk tolerance. An analogous result follows when we study the difference in effort across $\kappa$.

With respect to the percentage change in the certainty equivalent wealth, we see that the potential "efficiency" loss that arises from compensating the manager through the suboptimal $\alpha_{FB}$ is almost negligible when the manager is sufficiently risk-tolerant $(1/a = 24)$. However, in the standard situation where the manager is assumed to be more risk-averse than the investor this loss can rise up to 9%, even when $\kappa = 10$. Moreover, as $\kappa$ gets tighter, this difference gets substantially enhanced. Also note that in the reverse direction, when the constraint vanishes the third best scenario converges into the unconstrained second best scenario.

[11] $C_{TB}(\alpha)$ denotes the amount of end-of-period wealth that gives the constrained investor the same utility as (14).

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**Appendix A: Quadratic contracts**

In this section, we study the quadratic contracts proposed by Bhattacharya and Pfleiderer (1985). This type of contracts are interesting because they are known to elicit truthful information about the signal observed by the portfolio manager. Hence, the portfolio can be formed by the investor.

Assume the investor offers the manager a quadratic contract $(\gamma, F)$. Given the contract, the manager puts in effort and reports the signal to the investor. The investor incorporates this information $(\tilde{x}\,|\,y)$ and decides the optimal portfolio $\theta(y)$. Hence, the conditional payoffs for the investor and the manager are, respectively:[12] $\tilde{W}_a^q(y) = F - \gamma\left(\tilde{x}\,|\,y - M\right)^2$ and $\tilde{W}_b^q(y) = \gamma\left(\tilde{x}\,|\,y - M\right)^2 - F + \theta\tilde{x}\,|\,y$, where $M(y) = \frac{e}{1+e}y$ is the reported conditional mean of the risky asset, $\tilde{x}\,|\,y$.

According to Bhattacharya and Pfleiderer (1985), the manager's expected utility under the quadratic contract is given by

$$E\left[U_a\left(\tilde{W}_a^q\right)\right] = -\exp(-aF + V(a,e)) \times \left(1 - \frac{2a\gamma}{1+e}\right)^{-1/2}. \tag{A1}$$

[12] We will use the superscript $q$ to distinguish between linear and quadratic contracts.

In deriving this result, **Assumption (S4):** $\gamma < \frac{1+e}{2a}$, is necessary to guarantee the convergence of the expected utility integrals.[13] This assumption will play an important role when we compare linear and quadratic contracts.

From the appendix in Stoughton (1993) we obtain the investor's conditional expected utility as a function of his portfolio choice $\theta(y)$ and the conditional mean, $M$:

$$E\left[U_b\left(\tilde{W}_b^q(y)\right)\right] = -\left(1+\frac{2b\gamma}{1+e}\right)^{-1/2} \times \exp\left(bF + \frac{b^2\theta^2}{4(b\gamma+(1+e)/2)} - b\theta M\right). \qquad \text{(A2)}$$

In the public-information case, the investor maximizes (A2) with respect to $\theta \in \Re$ and then averages across the signal $y$. The result is the investor's ex ante unconstrained expected utility[14] as a function of $\gamma$ and $e$.

Under [BSS], the investor's optimal portfolio solves for $\theta^q(y) = \arg\max_\theta E\left[U_b\left(\tilde{W}_b^q(y)\right)\right]$ subject to $\kappa \geq \theta \geq -\kappa$. Like in the linear case, let $\lambda_l \geq 0$ (lower bound) and $\lambda_u \geq 0$ (upper bound) denote the corresponding Lagrangian multipliers, such that, at the optimal $(\theta^q + \kappa)\lambda_l = 0$ and $(\theta^q - \kappa)\lambda_u = 0$.

Define now the function $Q(\gamma) \equiv \left(\frac{2a\gamma}{1+e} + r\right)^{-1}$. Notice that, given assumption (S4), $Q(\gamma) > \alpha_{FB}$ for all $\gamma$. Conditional on $y$, there are three possible solutions: (i) If $\lambda_u = 0$ and $\lambda_l = -\frac{e}{1+e}b\left(y + \frac{\kappa a Q(\gamma)}{e}\right) > 0$, then short-selling is maximum and $\theta^q(y) = -\kappa$; (ii) if $\lambda_l = 0$ and $\lambda_u = \frac{e}{1+e}b\left(y - \frac{\kappa a Q(\gamma)}{e}\right) > 0$, the leverage is at the maximum and $\theta^q(y) = \kappa$. Otherwise, $\lambda_l = \lambda_u = 0$, and the optimal portfolio is $\theta^q(y) = \frac{e}{aQ(\gamma)}y$.

Writing the optimal portfolio as a function of the signal $y$, we have:

$$\theta^q(y) = \begin{cases} -\kappa & \text{if } y < -\frac{\kappa a Q(\gamma)}{e} \\ \frac{e}{aQ(\gamma)}y & \text{if } |y| \leq \frac{\kappa a Q(\gamma)}{e} \\ \kappa & \text{if } y > \frac{\kappa a Q(\gamma)}{e}. \end{cases} \qquad \text{(A3)}$$

The reader can verify that the optimal constrained portfolio for linear contract, (9), and the quadratic contract, (A3), coincide for $\alpha = Q(\gamma)$.

Plugging the portfolio choice (A3) in (A2) we obtain the following conditional expected utility for the constrained investor:

[13] The authors claim (Section 4, page 15) that "the distribution of wealth obtained by the agent when this inequality is violated is dominated by every distribution which can be obtained when the inequality is observed."

[14] Stoughton (1993), Proposition 2, equation (25).

$$E\left[U_b\left(\tilde{W}_b^q(y)\right)\right] = -\exp(aF/r)\times \tag{A4}$$

$$\left(1+\frac{2b\gamma}{1+e}\right)^{-1/2}\times\begin{cases}\exp\left(\frac{b}{aQ(\gamma)}\frac{e}{1+e}\kappa aQ(\gamma)\left(y+\frac{\kappa aQ(\gamma)}{2e}\right)\right) & \text{if } y<-\frac{\kappa aQ(\gamma)}{e}\\ \exp\left(-\frac{b}{aQ(\gamma)}\frac{e^2}{2(1+e)}y^2\right) & \text{if } |\,y|\leq\frac{\kappa aQ(\gamma)}{e}\\ \exp\left(-\frac{b}{aQ(\gamma)}\frac{e}{(1+e)}\kappa aQ(\gamma)\left(y-\frac{\kappa aQ(\gamma)}{2e}\right)\right) & \text{if } y>\frac{\kappa aQ(\gamma)}{e}.\end{cases}$$

We are now in a position to derive the investor's unconditional expected utility as a function of the contract $(\gamma, F)$ and effort. The result is presented in the following proposition whose proof follows trivially given (A4) and the proof of Proposition 1 in the Appendix B.

**Proposition 5** *Under [BSS], for a given quadratic contract $(\gamma, F)$, the expected utility function of the risk-averse investor is:*

$$E\left[U_b\left(\tilde{W}_b^q(\gamma,F,e\,|\,\kappa)\right)\right] = -exp(aF/r)\times\left(1+\frac{2b\gamma}{1+e}\right)^{-1/2}\times\left(g_\kappa(e\,|\,Q(\gamma))\right)^{\frac{b}{aQ(\gamma)}}.$$

Provided that the participation constraint is binding, the investor's expected utility becomes a function of $\gamma$ and $e$:

$$E\left[U_b\left(\tilde{W}_b^q(\gamma,e\,|\,\kappa)\right)\right] = -\exp(V(a,e)/r)\times$$
$$\left(1-\frac{2a\gamma}{1+e}\right)^{-b/(2a)}\times\left(1+\frac{2b\gamma}{1+e}\right)^{-1/2}\times\left(g_\kappa(e\,|\,Q(\gamma))\right)^{\frac{b}{aQ(\gamma)}}. \tag{A5}$$

At this point, we can compare linear and quadratic contracts *when the manager's effort decision is observed by the investor*, both under [BSS] and in the unconstrained case.

**Proposition 6** *Assume that the manager's effort decision is observable by the investor. Then, given (S4), the risk averse investor prefers the linear over the quadratic contract, both under [BSS] and in the unconstrained case.*

We are unable to analytically compare linear and quadratic contracts under moral hazard. So we resort to numerical methods. We assume that quadratic contacts induce truthful revelation even under [BSS]. Thus, in what follows, the investor's utility under quadratic contracts should be thought of as an upper bound. Furthermore, investor's utility under linear contracts are derived under the model where the manager (instead of the investor) forms the portfolio. The results would remain the same if we were to allow the investor to form the portfolio, and the investor commits to the schedule $\theta(y, e)$ which the manager forms in our model. This trivially induces truthful reporting of $(y, e)$. However, it may not be the optimal mechanism to induce truthful reporting under linear contracts.

Thus, the reported investor's utility under linear contracts should be thought of as a lower bound. To recapitulate, in what follows, we compare the highest possible investor's utility under quadratic contracts to the lowest possible investor's utility under linear contracts.

In the presence of moral hazard, the manager maximizes her expected utility (A1) with respect to effort given the contract $(\gamma, F)$. This yields the following first-order condition for the *quadratic second best effort*, $e^q_{SB}$:

$$V'(a, e^q_{SB}) = \frac{1}{2(1+e^q_{SB})}\left(1 - \frac{2a\gamma}{1+e^q_{SB}}\right)^{-1}\frac{2a\gamma}{1+e^q_{SB}}. \tag{A6}$$

Notice that, for the quadratic contract, *the manager's effort decision in increasing in* $\gamma$. Hence, the non-incentive result from linear contracts can be overcome by offering the manager a quadratic contract.

The investor will maximizes his expected utility (A5) subject to the manager's optimal effort decision (A6). Like in the linear case, we cannot solve analytically for the quadratic third best contract. We follow a numerical procedure similar to the analysis we used in Section 4.1.

We assume the same effort disutility function, $V(a, e) = ae^2$. Replacing this function in (A6) we obtain the following condition:

$$\gamma(a, e) = \frac{2e(1+e)^2}{4ae(1+e)+1}. \tag{A7}$$

The reader can easily verify that $\gamma(a, e) < \frac{1+e}{2a}$ hence satisfying assumption (S4). Notice that $\gamma(a, e)$ is decreasing in $a$.

We replace the later expression in (A5) and solve for the optimal third best effort as a function of the manager's risk aversion coefficient ($1/a \in \{3, 8, 15, 24\}$) and $\kappa = 1, 2, ..., 10$. The investor's risk tolerance is assumed to be $1/b = 24$. Plugging these values back into (A7) we obtain the third best values of $\gamma$. Like in the linear case, the plots (not shown here) of the expected utility as a function of $\gamma$ are always concave. The quadratic second and third best optimal effort expenditure, $\gamma$s and expected utility are reported in Table 2. We also report, for comparison, the corresponding linear values for effort and expected utility.

For all values of the manager's risk tolerance except the highest ($1/a = 1/b = 24$), the second best quadratic effort is higher than the linear effort. Inspite of this, the investor derives higher utility from linear contracts (except for $1/a = 3$ and $\kappa > 4$). This is because it is "cheaper" to induce effort through linear contracts. Moreover, and in general, when the short selling constraint gets tighter ($\kappa$ decreases) both levels of effort converge.

Like in Stoughton (1993), when the gap in risk tolerance coefficients between agent and principal is large enough (in our case, for $1/a = 3$), unconstrained, second best quadratic contracts dominate linear contracts. Interestingly, when the manager's constraint becomes tighter (concretely for $\kappa < 5$) the result reverses: *linear contracts dominate quadratic contracts.*

To gain more intuition about this result, Figure 2 shows, for four different values of $\kappa \in \{1, 10, 100, 1000\}$, the investor's percentage loss in certainty equivalent wealth (relative to the first best certainty equivalent wealth), as a function of his risk tolerance coefficient, when the manager is compensated with a quadratic contract. This is a measure of the efficiency loss induced by moral hazard relative to the public-information scenario. The lower right corner graph ($\kappa = 1000$) corresponds, in the limit, to the (unconstrained) second best convergence result (Figure 2, page 2022) reported in Stoughton (1993): the agency cost under quadratic contract drops off rapidly as a function of the principal's risk tolerance. However, when $\kappa$ is finite, increasing the manager's risk tolerance produces quite the opposite result: after an initial reduction (the more limited the lower $\kappa$ is), the efficiency loss from using quadratic contracts increases with the investor's risk tolerance.

**Appendix B: Proofs**

**The investor's unconditional expected utility.** Given her utility function and the definition of her conditional wealth in (2), the investor's (conditional) expected utility function can be written as a function of $M(\alpha)$ as follows:

$$E\left[U_b\left(\tilde{W}_b(y)\right)\right] = -\exp(aF/r) \times \exp\left(-\frac{e^2}{2(1+e)}y^2 M(\alpha)\right). \tag{B1}$$

The investor's unconditional expected utility, $E\left[U_b\left(\tilde{W}_b(\alpha, F, e)\right)\right] = \int_{-\infty}^{\infty} E\left[U_b\left(\tilde{W}_b(y)\right)\right] dF(y)$. The signal variable is normally distributed, $\tilde{y} \sim \mathcal{N}\left(0, \frac{1+e}{e}\right)$. Then, define the investor's unconditional exp $E\left[U_b\left(\tilde{W}_b(\alpha, F, e)\right)\right] = -\exp(aF/r) \times \left(\frac{e}{1+e}\right)^{1/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-(1/2)y^2 e\,\frac{1+eM(\alpha)}{1+e}\right) dy$. Substituting $s = y^2 e\,\frac{1+eM(\alpha)}{1+e}$ in the later equation and given the definition of $\Phi(\cdot)$ we arrive at equation (6).

**Proof of Proposition 1.** The manager's conditional expected utility is $E\left[U_a\left(\tilde{W}_a(y)\right)\right] =$

$$-\exp(-aF + V(a,e)) \times \begin{cases} \exp\left(\frac{e}{1+e}\kappa a\alpha\left(y + \frac{\kappa a\alpha}{2e}\right)\right) & \text{if } y < -\frac{\kappa a\alpha}{e} \\ \exp\left(-\frac{e^2}{2(1+e)}y^2\right) & \text{if } |y| \le \frac{\kappa a\alpha}{e} \\ \exp\left(-\frac{e}{(1+e)}\kappa a\alpha\left(y - \frac{\kappa a\alpha}{2e}\right)\right) & \text{if } y > \frac{\kappa a\alpha}{e}. \end{cases}$$

Taking the expectation across $y$ we obtain $E\left[U_a\left(\tilde{W}_a(\alpha, F, e\,|\,\kappa)\right)\right] = -\exp(-aF + V(a,e)) \times \left(\frac{e}{1+e}\right)^{1/2} \times \left[\exp\left(\frac{(\kappa a\alpha)^2}{2}\right)\int_{-\infty}^{-\frac{\kappa a\alpha}{e}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{e}{2(1+e)}(y-\kappa a\alpha)^2\right) dy + \int_{-\frac{\kappa a\alpha}{e}}^{\frac{\kappa a\alpha}{e}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{e}{2}\,y^2\right) dy + \exp\left(\frac{(\kappa a\alpha)^2}{2}\right)\int_{\frac{\kappa a\alpha}{e}}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{e}{2(1+e)}(y+\kappa a\alpha)^2\right) dy\right]$, the manager's unconditional expected utility. We propose the following change of variable: $s = \frac{e}{(1+e)}(y-\kappa a\alpha)^2$ if $y < -\frac{\kappa a\alpha}{e}$; $s = e(1+e)\,y^2$ if $|y| \le \frac{\kappa a\alpha}{e}$ and $s = \frac{e}{(1+e)}(y+\kappa a\alpha)^2$ if $y > \frac{\kappa a\alpha}{e}$. Replacing the new variable in the manager's unconditional expected utility and given the definition of $\Phi(\cdot)$ we arrive at (10).

The first derivative of $g_\kappa(e|\alpha)$ with respect to $e$ is:

$$g_\kappa'(e\,|\,\alpha) = -\frac{1}{2}\left(\frac{1}{1+e}\right)^{3/2} \times \Phi\left(\frac{(\kappa a\alpha)^2}{e}\right) < 0, \tag{B2}$$

for all $\alpha \in (0,1]$. Taking the second derivative with respect to $e$ we obtain:

$$g_\kappa''(e\,|\,a) = \frac{1}{2}\left(\frac{1}{1+e}\right)^{3/2}\left[\frac{3}{2}\left(\frac{1}{1+e}\right) \times \Phi\left(\frac{(\kappa a\alpha)^2}{e}\right) + \left(\frac{\kappa a\alpha}{e}\right)^2 \times \phi\left(\frac{(\kappa a\alpha)^2}{e}\right)\right] > 0. \tag{B3}$$

**Proof of Corollary 1.** First, we need the following lemma:

**Lemma 1** *For all* $0 < x < \infty$, $\phi(x) - \frac{1}{2}(1-\Phi(x)) > 0$.

**Proof:** For all $x > 0$, $\frac{1}{2}(1-\Phi(x)) = \frac{1}{\sqrt{2\pi}}\exp(-x/2)\,x^{-1/2} - \frac{1}{2}\int_x^\infty \frac{1}{\sqrt{2\pi}}\exp(-s/2)\,s^{-3/2}\,ds$. Therefore, $\phi(x) - \frac{1}{2}(1-\Phi(x)) = \frac{1}{2}\int_x^\infty \frac{1}{\sqrt{2\pi}}\exp(-s/2)\,s^{-3/2}\,ds > 0$.

Given the manager's expected utility in Proposition 1 the first part of the corollary will be proved if we can show that the function $g_\kappa(e\,|\,\alpha)$ is decreasing in $\alpha$. Given Lemma 1, $\frac{\partial}{\partial\alpha}g_\kappa(e\,|\,\alpha) = -2(\kappa a)^2\alpha\left[\phi\left(\frac{(\kappa a\alpha)^2}{e}(1+e)\right) - \frac{1}{2}\left(1-\Phi\left(\frac{(\kappa a\alpha)^2}{e}(1+e)\right)\right)\right] \times \exp\left(\frac{(\kappa a\alpha)^2}{2}\right) < 0$, for all $\alpha \in (0,1]$.

To prove the second part, we show that $\lim_{\kappa a\to\infty} g_\kappa(e\,|\,\alpha) = g(e)$. By definition, $\lim_{x\to\infty}\Phi(x) = 1$. Therefore, we need to show that

$$\lim_{x\to\infty}\left[\exp(x/2) \times \left(1-\Phi\left(x\,\frac{1+e}{e}\right)\right)\right] = 0. \tag{B4}$$

Let us re-write (B4) as $\lim_{x\to\infty}\frac{1-\Phi\left(x\,\frac{1+e}{e}\right)}{\exp(-x/2)}$. Both functions (exponential and $\Phi(\cdot)$) are continuous and differentiable. Taking the derivative of the numerator and the denominator with respect to $x$, the limit in (B4) is equal to $\lim_{x\to\infty}\frac{\exp(-x/e)}{x} = 0$.

**Proof of Proposition 2.** First, we prove the existence and uniqueness of $e_{TB}$. Let us call $\mathcal{J}_\kappa(e\,|\,\alpha) = V'(a,e) \times g_\kappa(e\,|\,\alpha) + g_\kappa'(e\,|\,\alpha)$, the first derivative of the manager's expected utility function with respect to $e$. The third best effort satisfies:

$$\mathcal{J}_\kappa(e_{TB}\mid\alpha) = 0, \tag{B5}$$

$$\mathcal{J}_\kappa'(e_{TB}\mid\alpha) > 0. \tag{B6}$$

Condition (B5) can be written like follows:

$$V'(a,e_{TB}) = -\frac{g_\kappa'}{g_\kappa}(e_{TB}\,|\,\alpha). \tag{B7}$$

For $\alpha = 0$, $e_{TB}(0) = 0$. Let us prove that the right-hand side term is monotonous decreasing in $e$ for all $\alpha \in (0, 1]$. Taking the derivative of this term with respect to $e$ and given (10) and equations (B2) and (B3) we get $g''_\kappa(e \mid \alpha) \times g_\kappa(e \mid \alpha) - (g'_\kappa(e \mid \alpha))^2 > \frac{1}{2}\left(\frac{1}{1+e}\right)^3 \times \Phi^2\left(\frac{(a\kappa\alpha)^2}{e}\right) > 0$. Thus, $-\frac{g'_\kappa}{g_\kappa}(e\,|\,\alpha)$ is (monotonous) decreasing in $e$ for all $\alpha \in (0, 1]$ with domain $(0, 1/2]$. By assumption, $V'(a, e) > 0$ for all $e > 0$. Hence, for any $\alpha \in (0, 1]$ there exists a unique $e_{TB}(\alpha) > 0$ that solves condition (B5).

Condition (B6) can be written as $V''(a, e) > -\frac{g'_\kappa}{g_\kappa}(e\,|\,\alpha) \times V'(a, e) - \frac{g''_\kappa}{g_\kappa}(e\,|\,\alpha)$. Since $-\frac{g'_\kappa}{g_\kappa}(e\,|\,\alpha) < \frac{1}{2(1+e)}$ and $\frac{g''_\kappa}{g_\kappa}(e\,|\,\alpha) \geq 0$ for all $\alpha \in [0, 1]$, then assumption (S3) implies (B6).

We prove next that $e_{SB} > e_{TB}(\alpha)$ for all $\alpha \in [0, 1]$. The case of $\alpha = 0$ is trivial since $e_{TB}(0) = 0 < e_{SB}$. For $\alpha > 0$, let us re-write the function $\mathcal{J}_\kappa(e\,|\,\alpha)$ as $\mathcal{J}_\kappa(e\,|\,\alpha) = \left[V'(a, e) - \frac{1}{2(1+e)}\right] \times g(e) \times \Phi\left(\frac{(\kappa a\alpha)^2}{e}\right) + V'(a, e) \times \exp\left(\frac{(\kappa a\alpha)^2}{2}\right) \times \left(1 - \Phi\left(\frac{(\kappa a\alpha)^2}{e}(1+e)\right)\right)$. Evaluating this function at the second best effort and given (5) we obtain $\mathcal{J}_\kappa(e_{SB} \mid \alpha) = V'(a, e_{SB}) \times \exp\left(\frac{(\kappa a\alpha)^2}{2}\right) \times \left(1 - \Phi\left(\frac{(\kappa a\alpha)^2}{e_{SB}}(1 + e_{SB})\right)\right) > 0$. This implies $E'\left[U_a\left(\tilde{W}_a(\alpha, F, e_{SB}|\kappa)\right)\right] = -\exp(-aF + V(a, e_{SB})) \times \mathcal{J}_\kappa(e_{SB}\,|\,\alpha) < 0$. Therefore, for the constrained manager, the marginal utility of effort at $e_{SB}$ is negative. Since $e_{TB}$ is unique and the function is continuous in $e$, given conditions (B5) and (B6), it follows that $e_{SB} > e_{TB}$.

Finally, given equation (B4), $\mathcal{J}_\kappa(e_{SB}\,|\,\alpha)$ tends to zero when $\kappa a$ tend to infinity.

**Proof of Corollary 2.** We know that $e_{TB}(0) = 0$. According to (B5), for any $\hat{\alpha} \in (0, 1]$, there exists $e_{TB}(\hat{\alpha}) > 0$ such that $\mathcal{J}_\kappa(e_{TB}\,|\,\hat{\alpha}) = 0$. The function $\mathcal{J}_\kappa$ is continuous and differentiable with respect to $(\alpha, e)$. Given (B6), the implicit function theorem allows us to solve "locally" the equation; that is, to express $e$ as a function of $\alpha$ in a neighborhood of $(\hat{\alpha}, e_{TB})$.

More formally: given $\hat{\alpha} \in (0, 1]$ there exists a function $e(\alpha)$,continuous and differentiable, and an open ball $B(\hat{\alpha})$, such that $e(\hat{\alpha}) = e_{TB}$ and $\mathcal{J}_\kappa(e(\alpha)\,|\,\alpha) = 0$ for all $\alpha \in B(\hat{\alpha})$. Taking the derivative of the last equation with respect to $\alpha$ and evaluated at $\hat{\alpha}$, $\frac{\partial}{\partial\alpha}e(\hat{\alpha}) = -\frac{\partial}{\partial\alpha}\mathcal{J}_\kappa(e_{TB}\,|\,\hat{\alpha}) \times \mathcal{J}'^{-1}_\kappa(e_{TB}\,|\,\hat{\alpha})$. From (B6), $\mathcal{J}'_\kappa(e_{TB}\,|\,\hat{\alpha}) > 0$. Therefore, the proposition will be proved if we show $\frac{\partial}{\partial\alpha}\mathcal{J}_\kappa(e_{TB}\,|\,\hat{\alpha}) = V'(a, e_{TB}) \times \frac{\partial}{\partial\alpha}g_\kappa(e_{TB}|\,\hat{\alpha}) + \frac{\partial}{\partial\alpha}g'_\kappa(e_{TB}|\,\hat{\alpha}) < 0$, for all $\hat{\alpha} \in (0, 1]$. From (S2), $V'(a, e_{TB}) > 0$ . From Corollary 1, $\frac{\partial}{\partial\alpha}g_\kappa(e_{TB}|\,\hat{\alpha}) < 0$. Finally, given equation (B3), $\frac{\partial}{\partial\alpha}g'_\kappa(e_{TB}|\,\hat{\alpha}) < 0$. Since the proof holds for any $\hat{\alpha} \in (0, 1]$, the Corollary is proved.

**Proof of Proposition 3.** Given the investor's indirect utility function in Section 4 the investor's unconditional expected utility will be $E\left[U_b\left(\tilde{W}_b(\alpha, F, e\,|\,\kappa)\right)\right] = -\exp(aF/r) \times \left(\frac{e}{1+e}\right)^{1/2} \times$

$$\Bigg[\exp\left(\frac{(\kappa a\alpha\, m(\alpha))^2}{2}\right)\int_{-\infty}^{-\frac{\kappa a\alpha}{e}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{e}{2(1+e)}(y - \kappa a\alpha\, m(\alpha))^2\right)dy$$

$$+ \int_{-\frac{\kappa a\alpha}{e}}^{\frac{\kappa a\alpha}{e}} \frac{1}{\sqrt{2\pi}}\exp\left(-(1/2)y^2 e\,\frac{1 + eM(\alpha)}{1+e}\right)dy +$$

$$\exp\left(\frac{(\kappa a\alpha\, m(\alpha))^2}{2}\right)\int_{\frac{\kappa a\alpha}{e}}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{e}{2(1+e)}(y + \kappa a\alpha\, m(\alpha))^2\right)dy\Bigg].$$

We propose the following change of variable: $s = \frac{e}{(1+e)}(y - \kappa a\alpha\, m(\alpha))^2$ if $y < -\frac{\kappa a\alpha}{e}$; $s = y^2 e\,\frac{1+eM(\alpha)}{1+e}$ if $|\,y| \leq \frac{\kappa a\alpha}{e}$ and $s = \frac{e}{(1+e)}(y + \kappa a\alpha\, m(\alpha))^2$ if $y > \frac{\kappa a\alpha}{e}$. Replacing the new variable in the investor's unconditional expected utility we obtain (13).

**Proof of Proposition 4.** First, we prove the results under the assumption of public information. The following Lemma shows that the first best split is (first-order) optimal in the absence of moral hazard:

**Lemma 2** *Given any effort* $e > 0$, $\frac{\partial}{\partial\,\alpha}E\left[U_b\left(\tilde{W}_b(\alpha, e\,|\,\kappa)\right)\right]\Big|_{\alpha=\alpha_{FB}} = 0.$

**Proof:** Given the definition (14), $\frac{\partial}{\partial\,\alpha}E\left[U_b\left(\tilde{W}_b(a, e\,|\,\kappa)\right)\right] = -\exp(V(a,e)/r) \times g_\kappa(e\,|\,a)^{1/r}$ $\times\left(\frac{1}{r}\,g_\kappa(e\,|\,\alpha)^{-1} \times \frac{\partial}{\partial\,\alpha}g_\kappa(e\,|\,\alpha) \times f_\kappa(\alpha,e) + \frac{\partial}{\partial\,\alpha}f_\kappa(\alpha,e)\right)$. Evaluating this equation at $\alpha_{FB}$:

$$\frac{\partial}{\partial\alpha}E\left[U_b\left(\tilde{W}_b(\alpha_{FB}, e\,|\,\kappa)\right)\right] = \qquad \text{(B8)}$$
$$-\exp(V(a,e)/r) \times g_\kappa(e\,|\,a_{FB})^{1/r} \times \left(\frac{1}{r}\,\frac{\partial}{\partial\,\alpha}g_\kappa(e\,|\,\alpha_{FB}) + \frac{\partial}{\partial\,\alpha}f_\kappa(\alpha_{FB}, e)\right).$$

Taking the derivative of $g_\kappa(e\,|\,\alpha)$ with respect to $\alpha$, $\frac{\partial}{\partial\,\alpha}g_\kappa(e\,|\,\alpha)\big|_{\alpha=\alpha_{SB}} = -2\,\frac{(\kappa a)^2}{1+r}\exp\left(\frac{1}{2}\left(\frac{\kappa a}{1+r}\right)^2\right)\times$ $\left[\phi\left(\frac{1+e}{e}\left(\frac{\kappa a}{1+r}\right)^2\right) - \frac{1}{2}\left(1 - \Phi\left(\frac{1+e}{e}\left(\frac{\kappa a}{1+r}\right)^2\right)\right)\right]$. Taking now the derivative of $f_\kappa(\alpha, e)$ with respect to $\alpha$, $\frac{\partial}{\partial\,\alpha}f_\kappa(\alpha_{SB}, e) = 2\frac{(\kappa a)^2}{r(1+r)}\exp\left(\frac{1}{2}\left(\frac{\kappa a}{1+r}\right)^2\right) \times \left[\phi\left(\frac{1+e}{e}\left(\frac{\kappa a}{1+r}\right)^2\right) - \frac{1}{2}\left(1 - \Phi\left(\frac{1+e}{e}\left(\frac{\kappa a}{1+r}\right)^2\right)\right)\right]$.
Replacing the two later expressions in (B8) the lemma is proved.

Evaluating (14) at $\alpha_{FB}$ yields the investor's expected utility function in the constrained public-information scenario as a function of effort:

$$E\left[U_b\left(\tilde{W}_b(\alpha_{FB}, e\,|\,\kappa)\right)\right] = -\exp(V(a,e)/r) \times g_\kappa(e\,|\,\alpha_{FB})^{(1+r)/r}. \qquad \text{(B9)}$$

Finally, taking the derivative of (B9) with respect to effort and making it equal to zero we obtain the following characterization of the constrained public-information effort $e_{CPI}$: $V'(a, e_{CPI}) = -(1+r)\frac{g'_\kappa}{g_\kappa}(e_{CPI}\,|\,\alpha_{FB})$. It is easy to show that when $\kappa \to \infty$ the later condition converges to condition (8) for the first best effort. Clearly, for any finite $\kappa$, $e_{CPI} < e_{FB}$.

We now prove the result under moral hazard. According to (B5), given the second best $\alpha_{SB} = \frac{1}{1+r} > 0$, there exists a unique $e_{TB}(\alpha_{SB}) > 0$ such that $\mathcal{J}_\kappa(e_{TB}\,|\,\alpha_{SB}) = 0$. Since the participation constraint is binding, $\alpha_{SB}$ will be optimal (necessary condition) only if $\frac{\partial}{\partial\,\alpha}E\left[U_b\left(\tilde{W}_b(\alpha, e(\alpha)\,|\,\kappa)\right)\right]\Big|_{\alpha=\alpha_{SB}} = 0$, where $e(\alpha)$ is, according to Corollary 2, a continuous and differentiable function, increasing in $\alpha$ with $e(\alpha_{SB}) = e_{TB}$.

We take the derivative of (14) with respect to $\alpha$,

$$\frac{\partial}{\partial\,\alpha}E\left[U_b\left(\tilde{W}_b(\alpha,e(\alpha)\,|\,\kappa)\right)\right]=$$
$$\frac{\partial}{\partial\,\alpha}E\left[U_b\left(\tilde{W}_b(\alpha,e)\,|\,\kappa)\right)\right]+\frac{\partial}{\partial\,e}E\left[U_b\left(\tilde{W}_b(\alpha,e)\,|\,\kappa)\right)\right]\times\frac{\partial}{\partial\,\alpha}e(\alpha).$$

Evaluating (14) at $\alpha_{SB}$, $E\left[U_b\left(\tilde{W}_b(\alpha_{SB},e\,|\,\kappa)\right)\right]=-\exp(V(a,e)/r)\times g_\kappa(e\,|\,\alpha_{SB})^{(1+r)/r}$. Taking the derivative of the latter expression with respect to $e$ and evaluating it at $e_{TB}$:

$$\frac{\partial}{\partial\,e}E\left[U_b\left(\tilde{W}_b(\alpha_{SB},e_{TB}\,|\,\kappa)\right)\right]=$$
$$-\frac{1}{r}\exp(V(a,e_{TB})/r)\times g_\kappa(e_{TB}\,|\,\alpha_{SB})^{1/r}\times\left[\mathcal{J}_\kappa(e_{TB}\,|\,\alpha_{SB})+rg'_\kappa(e_{TB}\,|\,\alpha_{SB})\right].$$

By definition, $\mathcal{J}_\kappa(e_{TB}\,|\,\alpha_{SB})=0$. From (B2), $g'_\kappa(e_{TB}\,|\,\alpha_{SB})<0$. Therefore, given Lemma 2, the definition of $g_\kappa(\cdot)$ in Proposition 1 and Corollary 2:

$$\frac{\partial}{\partial\,\alpha}E\left[U_b\left(\tilde{W}_b(\alpha_{SB},e(\alpha_{SB})\,|\,\kappa)\right)\right]=-\exp(V(a,e_{TB})/r)\times g_\kappa(e_{TB}\,|\,\alpha_{SB})^{1/r}\times g'_\kappa(e_{TB}\,|\,\alpha_{SB})\times\frac{\partial}{\partial\,\alpha}e(\alpha_{SB})>0.$$

Therefore, $\alpha_{SB}$ is suboptimal in the third best scenario.

**Proof of Proposition 6.** The measure used to compare both contracts is the investor's certainty equivalent wealth. Given the investor's utility function, $U_b(\tilde{W}_b)=-\exp(-b\tilde{W}_b)$, the certainty equivalent wealth of the expected utility $u$ is given by the inverse of this function, $C(u)=-\ln(-u)/b$. Clearly, for any two values of the investor's expected utility, $u_1$ and $u_2$, $u_1>u_2$ if and only if $C(u_1)>C(u_2)$.

Given Lemma 2, $\alpha_{FB}$ is optimal in the linear, constrained public-information case. Hence, the investor's expected utility is given by equation (B9) and the constrained, linear certainty equivalent wealth (net of disutility of effort) turns out to be $C_\kappa(e,\alpha_{FB})=\frac{a+b}{ab}\,(\text{-ln}g_k(e\,|\,\alpha_{FB}))$.

In the case of quadratic contracts, given (A5), the constrained, quadratic certainty equivalent (net) wealth is given by $C^q_\kappa(e,\gamma)=\frac{1}{2a}\ln\left(1-\frac{2a\gamma}{1+e}\right)+\frac{1}{2b}\ln\left(1+\frac{2b\gamma}{1+e}\right)-\frac{\ln g_k(e|Q(\gamma))}{aQ(\gamma)}$. Taking the first-order Taylor expansion of the logarithmic function, we can rewrite the later expression as $C^q_\kappa(e,\gamma)\approx\frac{1}{aQ(\gamma)}\,(\text{-ln}g_k(e\,|\,Q(\gamma)))$.

From Proposition 1 and Corollary 1 we know that $g_k(e\,|\,\alpha)$ is decreasing in $\alpha$ and $e$ and bounded below one. Moreover, given equation (B2) in the Appendix B, $\frac{\partial}{\partial\alpha}\,g'_\kappa(e\,|\,\alpha)<0$. Since, by definition, $Q(\gamma)>\alpha_{FB}$ then $|g_k(e\,|\,Q(\gamma))|<|g_k(e\,|\,\alpha_{FB})|$ *for any* $e$ *and* $\gamma$. Therefore, given the definition of $Q(\gamma)$, we can write $C_\kappa(e,\alpha)-C^q_\kappa(e,\gamma)>\left(1-\frac{2a\gamma}{1+e}\right)\left(-\frac{\ln g_k(e\,|\,\alpha_{FB})}{a}\right)$, *for any* $\gamma$*,* $\alpha$ *and* $e$*.* It is now straightforward to see that the right-hand term in the later expression is strictly positive if and only if assumption (S4) holds.

Notice that the later proof holds for any $\kappa$. It is trivial to prove that the same result follows in the unconstrained scenario when $\kappa\to\infty$.

Table 1: **Optimal third best values of $\alpha$ and comparative statics with the first best for $1/b = 24$.**

$\Delta\alpha/\alpha$ and $\Delta e/e$ represent, respectively, the (percentage) change in the investor's optimal contract and the manager's effort expenditure when the later is offered the (sub-optimal) first best split $\alpha_{FB}$ in the constrained, third best scenario. $\Delta C/C$, can be interpreted as the net return that would compensate the investor for the lower utility of the suboptimal share $\alpha_{FB}$ in the third best scenario. The manager's disutility function of effort is assumed to be $V(a,e) = ae^2$. First best values $\alpha_{FB}$ are reported in parenthesis.

| | Value of the short-selling constraint $\kappa$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Manager's risk tolerance $1/a = 3$ ($\alpha_{FB} = 0.11$) | | | | | | | | | | |
| $\alpha_{TB}$ | 0.43 | 0.35 | 0.31 | 0.28 | 0.25 | 0.24 | 0.22 | 0.21 | 0.20 | 0.19 |
| $\Delta\alpha/\alpha$ | 287 | 215 | 179 | 152 | 125 | 116 | 98.0 | 89.0 | 80.0 | 70.9 |
| $\Delta e/e$ | 128 | 96.9 | 80.4 | 68.1 | 56.3 | 51.3 | 43.3 | 38.8 | 34.4 | 30.2 |
| $\Delta C/C$ | 29.0 | 22.8 | 19.3 | 16.8 | 14.9 | 13.4 | 12.1 | 11.0 | 10.0 | 09.1 |
| Manager's risk tolerance $1/a = 8$ ($\alpha_{FB} = 0.25$) | | | | | | | | | | |
| $\alpha_{TB}$ | 0.61 | 0.53 | 0.48 | 0.46 | 0.43 | 0.42 | 0.40 | 0.39 | 0.38 | 0.38 |
| $\Delta\alpha/\alpha$ | 144 | 112 | 92.0 | 84.0 | 72.0 | 68.0 | 60.0 | 56.0 | 52.0 | 52.0 |
| $\Delta e/e$ | 69.8 | 54.0 | 44.5 | 40.3 | 34.7 | 32.5 | 28.7 | 26.7 | 24.7 | 24.4 |
| $\Delta C/C$ | 13.1 | 10.1 | 08.5 | 07.5 | 06.7 | 06.2 | 05.7 | 05.3 | 05.0 | 04.7 |
| Manager's risk tolerance $1/a = 15$ ($\alpha_{FB} = 0.38$) | | | | | | | | | | |
| $\alpha_{TB}$ | 0.72 | 0.65 | 0.60 | 0.58 | 0.56 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 |
| $\Delta\alpha/\alpha$ | 87.2 | 69.0 | 56.0 | 50.8 | 45.6 | 40.4 | 37.8 | 35.2 | 32.6 | 30.0 |
| $\Delta e/e$ | 44.0 | 34.2 | 27.7 | 24.9 | 22.3 | 19.8 | 18.5 | 17.2 | 15.9 | 14.6 |
| $\Delta C/C$ | 06.7 | 05.0 | 04.1 | 03.5 | 03.1 | 02.8 | 02.6 | 02.4 | 02.3 | 02.1 |
| Manager's risk tolerance $1/a = 24$ ($\alpha_{FB} = 0.50$) | | | | | | | | | | |
| $\alpha_{TB}$ | 0.79 | 0.73 | 0.69 | 0.68 | 0.65 | 0.63 | 0.62 | 0.61 | 0.61 | 0.60 |
| $\Delta\alpha/\alpha$ | 58.0 | 46.0 | 38.0 | 36.0 | 30.0 | 26.0 | 24.0 | 22.0 | 22.0 | 20.0 |
| $\Delta e/e$ | 30.0 | 23.22 | 18.9 | 17.7 | 14.8 | 12.8 | 11.8 | 10.8 | 10.7 | 09.8 |
| $\Delta C/C$ | 03.7 | 02.7 | 02.2 | 01.8 | 01.6 | 01.4 | 01.3 | 01.2 | 01.1 | 01.0 |

Table 2: **Optimal values of $\gamma$, effort expenditure and expected utility for** $1/b = 24$. The manager's disutility function of effort is assumed to be $V(a,e) = ae^2$. The superscripts $Q$ and $L$ denote quadratic and linear case, respectively. The second best values (SB) are reported in the last row.

| | Manager's risk tolerance $1/a = 3$ | | | | | Manager's risk tolerance $1/a = 8$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\kappa$ | $\gamma_{TB}$ | $e^Q_{TB}$ | $e^L_{TB}$ | $EUtility^Q$ | $EUtility^L$ | $\gamma_{TB}$ | $e^Q_{TB}$ | $e^L_{TB}$ | $EUtility^Q$ | $EUtility^L$ |
| 1 | 0.34 | 0.16 | 0.16 | -0.991087 | -0.990022 | 0.48 | 0.19 | 0.20 | -0.990211 | -0.989047 |
| 2 | 0.55 | 0.25 | 0.22 | -0.979621 | -0.978200 | 0.77 | 0.28 | 0.28 | -0.977943 | -0.975501 |
| 3 | 0.71 | 0.32 | 0.26 | -0.967645 | -0.966392 | 1.03 | 0.35 | 0.33 | -0.965316 | -0.961540 |
| 4 | 0.85 | 0.38 | 0.28 | -0.955652 | -0.954970 | 1.26 | 0.41 | 0.38 | -0.9528 | -0.947647 |
| 5 | 0.98 | 0.44 | 0.30 | -0.943858 | -0.944080 | 1.46 | 0.46 | 0.41 | -0.940589 | -0.934030 |
| 6 | 1.10 | 0.49 | 0.32 | -0.932377 | -0.933765 | 1.67 | 0.51 | 0.45 | -0.928777 | -0.920793 |
| 7 | 1.21 | 0.54 | 0.33 | -0.92127 | -0.924058 | 1.85 | 0.55 | 0.48 | -0.917407 | -0.907998 |
| 8 | 1.32 | 0.59 | 0.35 | -0.91057 | -0.914938 | 2.03 | 0.59 | 0.50 | -0.906501 | -0.895667 |
| 9 | 1.41 | 0.63 | 0.36 | -0.900294 | -0.906402 | 2.21 | 0.63 | 0.53 | -0.89606 | -0.883818 |
| 10 | 1.52 | 0.68 | 0.37 | -0.890446 | -0.898428 | 2.39 | 0.67 | 0.56 | -0.886083 | -0.872449 |
| SB | 3.59 | 1.76 | 0.50 | -0.713478 | -0.804401 | 7.29 | 1.65 | 1.00 | -0.704506 | -0.656763 |
| | Manager's risk tolerance $1/a = 15$ | | | | | Manager's risk tolerance $1/a = 24$ | | | | |
| $\kappa$ | $\gamma_{TB}$ | $e^Q_{TB}$ | $e^L_{TB}$ | $EUtility^Q$ | $EUtility^L$ | $\gamma_{TB}$ | $e^Q_{TB}$ | $e^L_{TB}$ | $EUtility^Q$ | $EUtility^L$ |
| 1 | 0.57 | 0.21 | 0.22 | -0.98981 | -0.988605 | 0.62 | 0.22 | 0.23 | -0.989596 | -0.988357 |
| 2 | 0.91 | 0.30 | 0.31 | -0.977114 | -0.974225 | 0.99 | 0.31 | 0.33 | -0.976654 | -0.973472 |
| 3 | 1.22 | 0.37 | 0.37 | -0.964085 | -0.959209 | 1.33 | 0.38 | 0.40 | -0.963387 | -0.957794 |
| 4 | 1.51 | 0.43 | 0.43 | -0.951195 | -0.944096 | 1.65 | 0.44 | 0.46 | -0.950262 | -0.941906 |
| 5 | 1.76 | 0.48 | 0.47 | -0.938635 | -0.929127 | 2.00 | 0.50 | 0.51 | -0.937472 | -0.926050 |
| 6 | 2.04 | 0.53 | 0.51 | -0.926493 | -0.914429 | 2.24 | 0.54 | 0.55 | -0.925109 | -0.910391 |
| 7 | 2.26 | 0.57 | 0.55 | -0.914813 | -0.900075 | 2.57 | 0.59 | 0.59 | -0.913206 | -0.895004 |
| 8 | 2.50 | 0.61 | 0.58 | -0.90361 | -0.886108 | 2.85 | 0.63 | 0.63 | -0.901786 | -0.879943 |
| 9 | 2.75 | 0.65 | 0.61 | -0.892884 | -0.872550 | 3.14 | 0.67 | 0.67 | -0.890849 | -0.865238 |
| 10 | 3.00 | 0.69 | 0.64 | -0.882634 | -0.859416 | 3.37 | 0.70 | 0.70 | -0.880389 | -0.850908 |
| SB | 10.78 | 1.66 | 1.50 | -0.689056 | -0.521662 | 14.01 | 1.698 | 2.00 | -0.675152 | -0.393787 |

Figure 1: Investor's expected utility as a function of $\alpha$ for different values of $\kappa$. The manager and the investor are assumed to have the same risk tolerance coefficient $1/a = 1/b = 8$. The vertical lines denote the corresponding optimal third best alpha. The first best alpha is equal to 0.50.

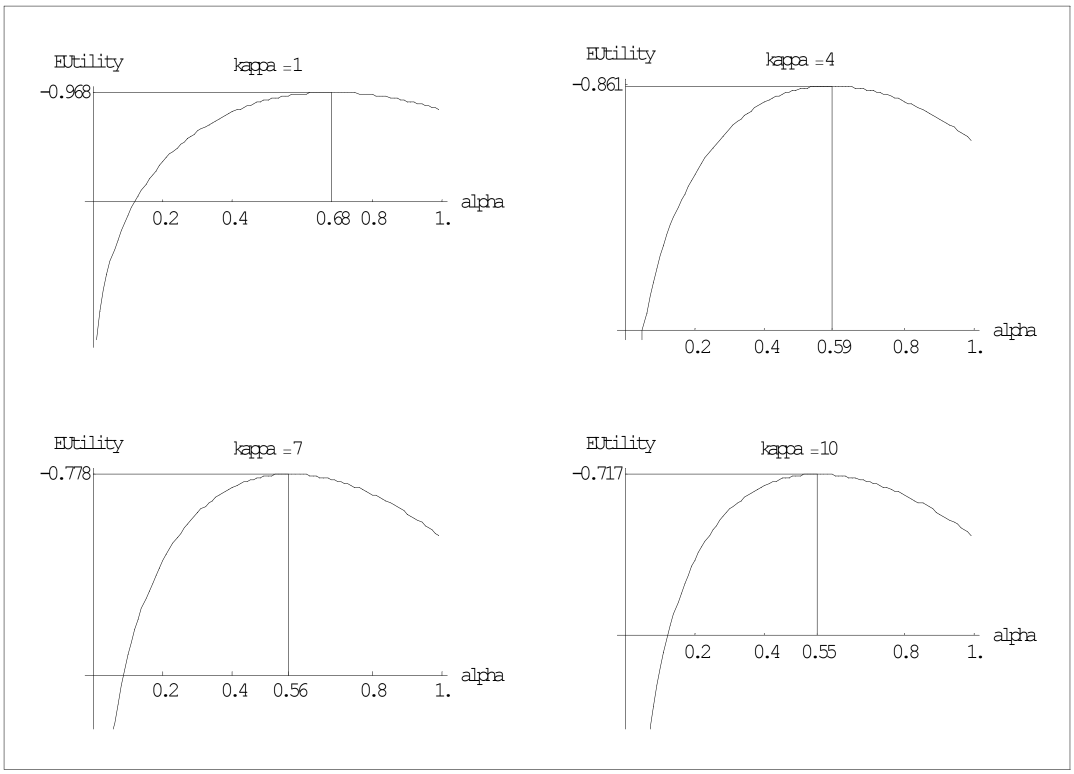

Figure 2: The investor's percentage certainty equivalent loss, $\Delta C/C$,(relative to the first best certainty equivalent wealth), as a function of his risk-tolerance coefficient $1/b$, when the manager is offered a quadratic contract.